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# Computer-Aided Education: Learning through Discovery

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**Abstract.** The computer program “Discoverer” is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. In this paper the authors discuss the use of “Discoverer” in the area of teaching and learning geometry with the help of a computer. The computer program “Discoverer” fills a gap in the existing set of educational tools. It provides the possibility the students easily to discover new theorems in Euclidean geometry. By using the principles of “Discoverer”, a number of similar computer programs could be created for the areas of high school teaching: physics, chemistry, biology, and so on. These computer programs could be considered as a tool for activation of the interest of the students. We call the use of “Discoverer” for educational purposes “learning through discovery”. We may consider the “learning through discovery” as a new important direction within the “learning through inquiry”.

## 1. INTRODUCTION

Dynamic geometry systems like GeoGebra or C.a.R., accompanied by computer algebra systems such as Maple or Mathematica, have highly influenced high school and university education in the area of Euclidean geometry. See e.g (Botana & Valcarce, 2002), (Hašek, 2013). But still there is a missing tool – a computer program able to discover new theorems in Euclidean geometry. The computer program “Discoverer”, created by the authors of this paper, fills the gap in the existing set of tools. The results of the “Discoverer” are expressed in natural language.

The computer program “Discoverer” is an artificial intelligence system for discovery of new knowledge, able easily to produce new theorem in Euclidean geometry.

As far as the authors know, the “Discoverer” is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. The use of “Discoverer” now begins, so that we could expect many remarkable results in the future.

In this paper we discuss the use of “Discoverer” in high school and university education. See also (Grozdev and Dekov, 2013a,b, 2014a,b). The “Discoverer” provides the possibility the students easily to discover new theorems, and by the help of the other tools, to clarify the theorems. During the last years many researchers and educators work actively on the inquiry-based teaching and learning methods in science and mathematics in primary and secondary schools. This direction is known as the “learning through inquiry”. See e.g. (Fibonacci Project). We call the use of “Discoverer” for educational purposes “learning through discovery”. We may consider the “learning through discovery” as a new important direction within the “learning through inquiry”.

We could use “Discoverer” for many purposes in Euclidean Geometry. E.g., we could use the “Discoverer” to find new compass-and-ruler solutions of classical construction problems, like the Malfatti circles problem. See (Grozdev & Dekov, 2013a). Note that the prototype of the “Discoverer” has discovered a new compass-and-ruler construction for the point “Yff Center of Congruence”, and this construction is already quoted in the Wikipedia. See (Wikipedia, Yff Center of Congruence).

In this paper we illustrate the use of the “Discoverer” in one of the Euclidean geometry topics. The selected topic is the “Investigation of the reflections of remarkable points in remarkable points in the geometry of triangle”. The “Discoverer” is written by using the programming language PHP, so that it produces HTML-files. Clearly, in many other topics of the Euclidean geometry we have to use other abilities of the “Discoverer”. The “Discoverer” is written by using the programming language PHP, so that it produces HTML-files. We enclose to this paper HTML-files, produced by the “Discoverer”.

## **2 MATHEMATICAL INVESTIGATION BY MEANS OF A COMPUTER**

We proceed as follows. First, we select from the database of the “Discoverer” a few remarkable points of the triangle. In this example we select the four classical remarkable points, namely the centroid, incenter, circumcenter and orthocenter. As a next step, the “Discoverer” produces a list of all points generated from the selected points by using the operation of the “Discoverer” named the “Reflection of a Point in a Point”. The produced list is enclosed to this paper. See the enclosed files, the file “1\_List\_P.php.htm” from the folder “Reflections 1-4”.

Next, we want to select the new points from the list of the produced points, that is, the points which are not investigated in the literature. In order to this, the “Discoverer” compares the points of the produced list with the points available in (Kimberling). The Kimberling’s Encyclopedia of Triangle Centers (Kimberling) contains the most complete list of remarkable points, currently available in the literature. The encyclopedia describes more than 5000 remarkable points. Hence, we may expect that if a remarkable point is not included in the ETC, most probably this point is not available in the literature.

The computer program “Discoverer” divides the list in the file “1\_List\_P.php.htm” into two lists: the list “List K” of points available in (Kimberling), and the list “List D” of points which are not available in (Kimberling). We see that “List D” contains only one point, namely the point “Reflection of the Incenter in the Orthocenter.” We may conclude that almost all reflections of the classical remarkable points in the classical remarkable points are studied in the literature. The only exception is the point named “Reflection of the Incenter in the Orthocenter”.

Next, we want to find theorems about the new point “Reflection of the Incenter in the Orthocenter”. Since this remarkable point is not studied in the literature, we may expect that the theorems about this point are new theorems. Now we use the ability of the “Discoverer” to discover new theorem about a selected point in the geometry of the triangle. The computer program “Discoverer” discovers theorems related to this new point. The list of the discovered theorems is enclosed to this paper. See the enclosed file “Theorems.htm”. The list contains nineteen theorems.

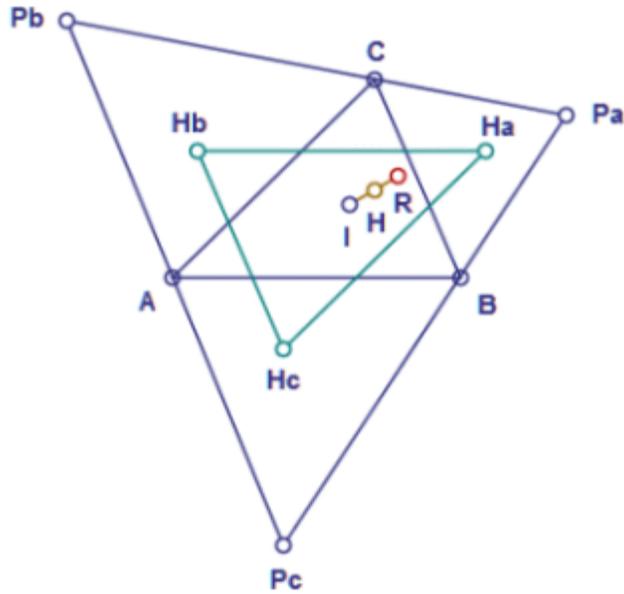
### 3 THE USE OF A DYNAMIC GEOMETRY SYSTEM

In this section we illustrate the theorems no 1, 4, 7, 8, 10, 11, 16 and 19 in the list of theorems, given in the file “Theorems.htm”. We use the dynamic geometry system C.a.R., created by Rene Grothmann of Germany. The computer program C.a.R. is available for download. It is free and open source.

Recall that the de Longchamps point of a triangle is the reflection of the orthocenter of the triangle in the circumcenter of the triangle (See Wikipedia, de Longchamps point).

**Theorem 1.** The Reflection of the Incenter in the Orthocenter coincides with the de Longchamps Point of the Triangle of the Orthocenters of the Anticevian Corner Triangles of the Incenter.

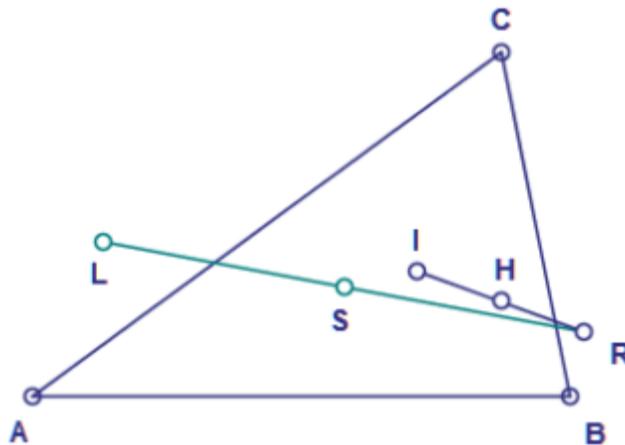
Figure 1 illustrates the theorem. In fig.1,  $I$  is the incenter,  $H$  is the orthocenter,  $R$  is the reflection of the incenter in the orthocenter,  $PaPbPc$  is the anticevian triangle of the incenter (that is, the excentral triangle of  $\triangle ABC$ ),  $Ha$ ,  $Hb$  and  $Hc$  are the orthocenters of triangles  $PaBC$ ,  $PbCA$  and  $PcAB$ , respectively. Then point  $R$  coincides with the de Longchamps point of triangle  $HaHbHc$ .



**Fig. 1.**

Recall that the Spieker center of a triangle is the incenter of the medial triangle of the triangle (See e.g. Wikipedia, Spieker center).

**Theorem 2.** The Reflection of the Incenter in the Orthocenter coincides with the Reflection of the de Longchamps Point in the Spieker Center.

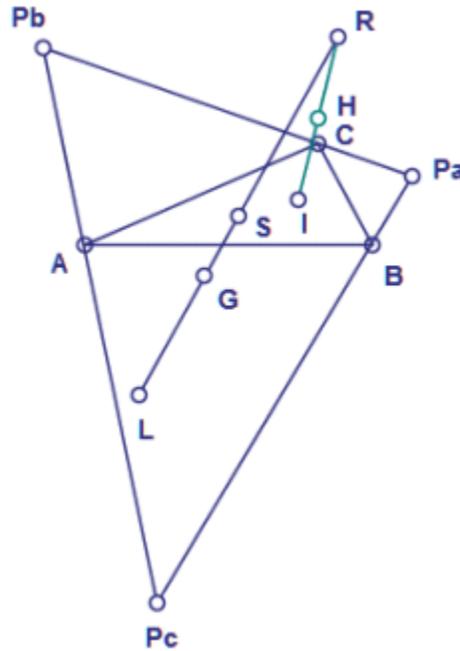


**Fig. 2.**

Figure 2 illustrates the theorem. In fig.2,  $I$  is the incenter,  $H$  is the orthocenter,  $R$  is the reflection of the incenter in the orthocenter,  $L$  is the de Longchamps point,  $S$  is the Spieker center. Then point  $R$  coincides with the reflection of the de Longchamps point in the Spieker center.

**Theorem 3.** The Reflection of the Incenter in the Orthocenter coincides with the Harmonic Conjugate of the Centroid of the Excentral Triangle with respect to the Spieker Center and the de Longchamps Point.

Figure 3 illustrates the theorem. In fig.3,  $I$  is the incenter,  $H$  is the orthocenter,  $R$  is the reflection of the incenter in the orthocenter,  $L$  is the de Longchamps point,  $S$  is the Spieker center,  $PaPbPc$  is the excentral triangle of  $\triangle ABC$ ,  $G$  is the centroid of triangle  $PaPbPc$ . Then point  $R$  coincides with the harmonic conjugate of the centroid of the excentral triangle with respect to the Spieker center and the de Longchamps point.



**Fig. 3.**

**Theorem 4.** The Reflection of the Incenter in the Orthocenter coincides with the Prasolov Product of the Centroid and the Spieker Center of the Euler Triangle.

Given triangle  $ABC$  and points  $P$  and  $Q$ . Denote by  $MaMbMc$  the cevian triangle of  $P$ . Denote by  $Pa$  the reflection of point  $Ma$  in point  $Q$ , by  $Pb$  the reflection of point  $Mb$  in point  $Q$ , and by  $Pc$  the reflection of point  $Mc$  in point  $Q$ . If the lines  $PaA$ ,  $PbB$  and  $PcC$  concur in a point, we say that the point of concurrence is the *Prasolov product of points P and Q*. This definition generalizes the definition of the Prasolov point. For the definition of Prasolov point, see e.g. (Weisstein, Prasolov point).

Figure 4 illustrates the theorem. In fig.4,  $I$  is the incenter,  $H$  is the orthocenter,  $R$  is the reflection of the incenter in the orthocenter,  $MaMbMc$  is the medial triangle of  $\triangle ABC$ ,  $S$  is the Spieker center of the Euler triangle of  $\triangle ABC$ ,  $Pa$  is the reflection of point  $Ma$  in point  $S$ ,  $Pb$  is the reflection of point  $Mb$  in point  $S$ , and  $Pc$  is the reflection of point  $Mc$  in point  $S$ . Then the lines  $PaA$ ,  $PbB$  and  $PcC$  concur in point  $R$ . (The lines are not drawn in the figure).

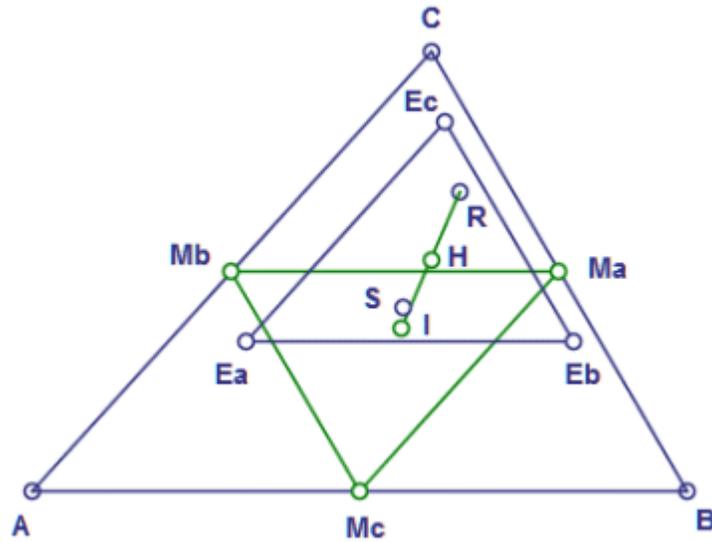


Fig. 4.

**Theorem 5.** The Reflection of the Incenter in the Orthocenter coincides with the Center of the Taylor Circle of the Hexyl Triangle of the Antimedial Triangle.

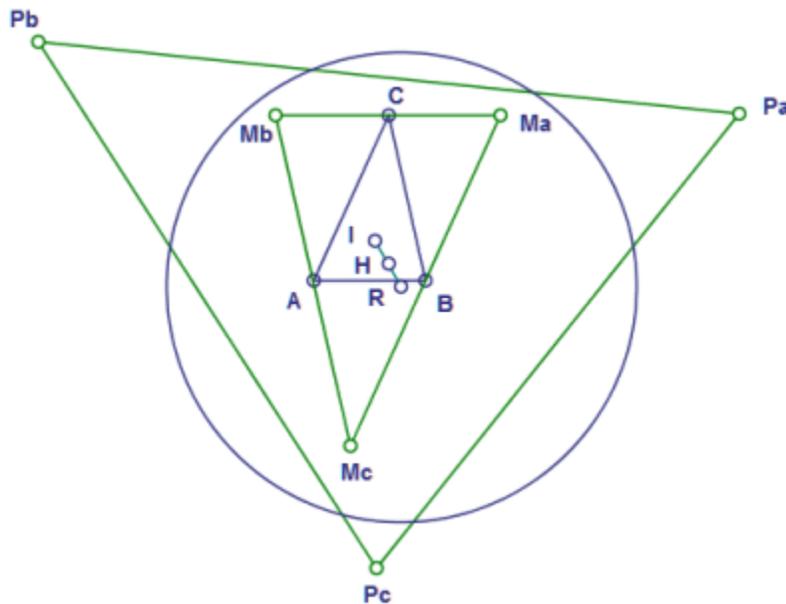


Fig. 5.

Recall the definitions of the hexyl triangle and the Taylor circle.

Given the excentral triangle  $J_aJ_bJ_c$  of  $\triangle ABC$ , define  $H_a$  as the point in which the perpendicular to  $AB$  through the excenter  $J_b$  meets the perpendicular to  $AC$  through the excenter  $J_c$ , and similarly define  $H_b$  and  $H_c$ . Then  $\triangle H_aH_bH_c$  is known as the hexyl triangle of  $ABC$  (See Weisstein, Hexyl Triangle).

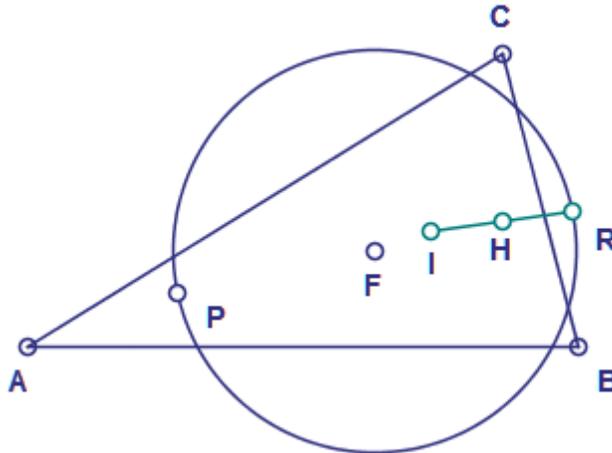
From each of the feet  $Ha$ ,  $Hb$  and  $Hc$  of the altitudes of  $\triangle ABC$  drop a pair of perpendiculars on the other two sides of the triangle. Then the feet of these six perpendiculars lie on a circle known as the Taylor circle. (See Weisstein, Taylor Circle).

Figure 5 illustrates the theorem. In fig.5,  $I$  is the incenter,  $H$  is the orthocenter,  $R$  is the reflection of the incenter in the orthocenter,  $MaMbMc$  is the antimedial triangle of  $\triangle ABC$ , triangle  $PaPbPc$  is the hexyl triangle of triangle  $MaMbMc$ , and  $c$  is the Taylor circle of triangle  $PaPbPc$ . Then  $R$  is the center of the Taylor circle  $c$ .

**Theorem 6.** The Reflection of the Incenter in the Orthocenter lies on the Circle having center at the Center of the Fuhrmann Circle and passing through the Bevan Point.

For the definitions of the Bevan point and the Fuhrmann circle we refer the reader to (Weisstein, Bevan Point),(Weisstein, Fuhrmann Circle).

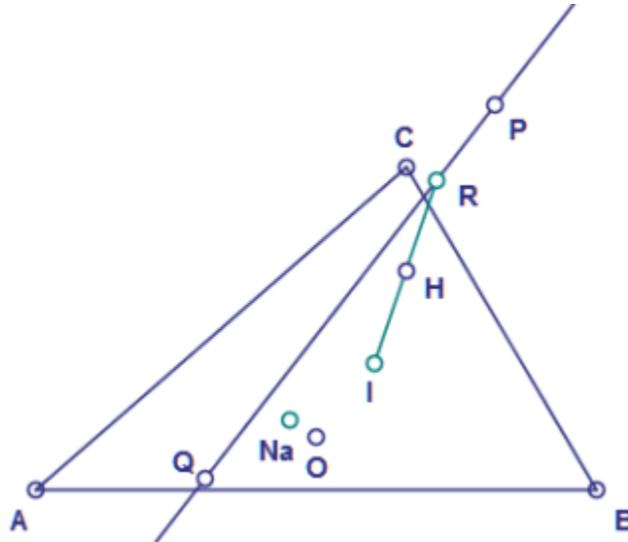
Figure 6 illustrates the theorem. In fig.6,  $I$  is the incenter,  $H$  is the orthocenter,  $R$  is the reflection of the incenter in the orthocenter,  $P$  is the Bevan point and  $F$  is the center of the of the Fuhrmann circle. Then point  $R$  lies on the circle having center at the center of the Fuhrmann circle and passing through the Bevan point.



**Fig. 6.**

**Theorem 7.** The Reflection of the Incenter in the Orthocenter lies on the Line through the Reflection of the Circumcenter in the Orthocenter and the Reflection of the Incenter in the Nagel Point.

Figure 7 illustrates the theorem. In fig.7,  $I$  is the incenter,  $H$  is the orthocenter,  $R$  is the reflection of the incenter in the orthocenter,  $O$  is the circumcenter,  $P$  is the reflection of the circumcenter in the orthocenter,  $Na$  is the Nagel point and  $Q$  is the reflection of the incenter in the Nagel point. Then point  $R$  lies on the line  $PQ$ .

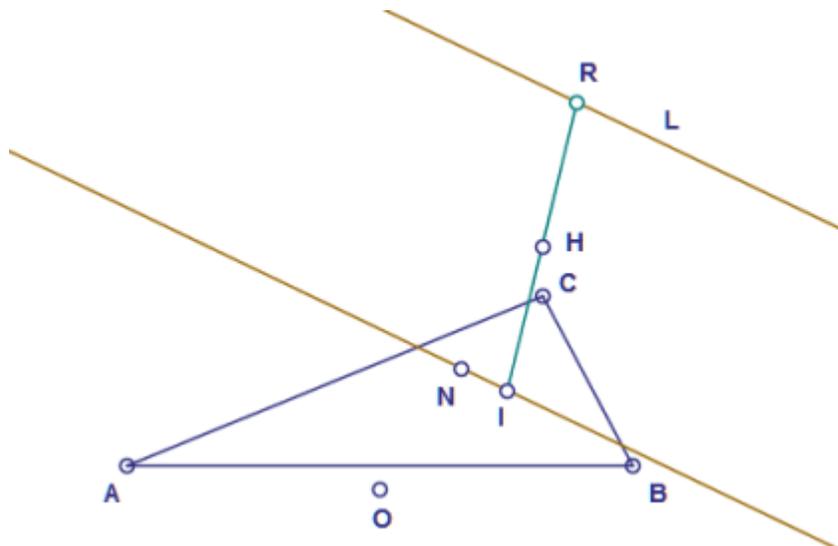


**Fig. 7.**

Recall that the Feuerbach line is the line passing through the incenter and the nine-point center.

**Theorem 8.** The Reflection of the Incenter in the Orthocenter lies on the Image of the Feuerbach Line under the Homothety with Center the Circumcenter and Ratio 3.

Figure 8 illustrates the theorem. In fig.8,  $I$  is the incenter,  $H$  is the orthocenter,  $R$  is the reflection of the incenter in the orthocenter,  $O$  is the circumcenter,  $N$  is the nine-point center,  $L$  is the image of the Feuerbach line under the homothety with center the circumcenter and ratio 3. Then point  $R$  lies on line  $L$ .



**Fig. 8.**

#### 4. THE USE OF A COMPUTER ALGEBRA SYSTEM

The proofs of the above theorems, as well as the other theorems from the enclosed files, would be a nice exercise for students. The teachers and professors could use theorems, discovered by “Discoverer” in order to variegate their lectures.

As we have mentioned, the point “Reflection of the Incenter in the Orthocenter” is not available in (Kimberling). If we want to submit this point for publication in (Kimberling), we have to submit the barycentric coordinates of the point. We could use a computer algebra system like Maple. We denote by  $a$ ,  $b$  and  $c$  the side lengths of triangle  $ABC$ ,  $a = BC$ ,  $b = CA$  and  $c = AB$ . The barycentric coordinates of the known remarkable points are available in (Kimberling). We proceed as follows. We write the barycentric coordinates of the incenter as  $(u_1, v_1, w_1)$ , and of the orthocenter as  $(u_2, v_2, w_2)$ , and we find the normalized barycentric coordinates of these points. Next, we calculate the reflection of the incenter in the orthocenter. Finally, we simplify the barycentric coordinates. The Maple file with the calculation of the barycentric coordinates of the point is enclosed to this paper. Below we give the Maple commands.

```

u1:=a;v1:=b;w1:=c;
u2:=1/(b^2+c^2-a^2);v2:=1/(c^2+a^2-b^2);w2:=1/(a^2+b^2-c^2);
u:=u1/(u1+v1+w1);v:=v1/(u1+v1+w1);w:=w1/(u1+v1+w1);
p:=u2/(u2+v2+w2);q:=v2/(u2+v2+w2);r:=w2/(u2+v2+w2);
x:=(p-q-r)*u+2*p*(v+w):x:=simplify(x);
y:=(q-r-p)*v+2*q*(w+u):y:=simplify(y);
z:=(r-p-q)*w+2*r*(u+v):z:=simplify(z);
Q_x:=denom(x);
x:=x*Q_x;
y:=y*Q_x;
z:=z*Q_x;

```

The first baricentric coordinate of the reflection of the incenter in the orthocenter is as follows:

$$x = 2b^4 + 2c^4 - 3a^4 + a^3b + a^3c - ab^3 - ac^3 + a^2b^2 + a^2c^2 - 4b^2c^2 + ab^2c + abc^2 - 2a^2bc.$$

Theorem 2 could be proved as follows. We use barycentric coordinates. As a first step, we find the normalized barycentric coordinates of the incenter and the orthocenter and then we find the reflection  $P$  of the incenter in the orthocenter. As a second step, we find the normalized barycentric coordinates of the de Longchamps point and the Spieker center and then we find the reflection  $Q$  of the de Longchamps point in the Spieker center. As a third step, we compare the barycentric coordinates of points  $P$  and  $Q$ , and we conclude that they coincide. This proves the theorem.

If we want to avoid calculations by hand in the above proof, we may use a computer algebra system like Maple. The Maple file with the proof of theorem 2 is enclosed to this paper. Below we give the Maple commands (copy and paste in a Maple file):

```

u1:=a;v1:=b;w1:=c;
u:=u1/(u1+v1+w1);v:=v1/(u1+v1+w1);w:=w1/(u1+v1+w1);
u2:=1/(b^2+c^2-a^2);v2:=1/(c^2+a^2-b^2);w2:=1/(a^2+b^2-c^2);
p:=u2/(u2+v2+w2);p:=simplify(p);
q:=v2/(u2+v2+w2);q:=simplify(q);
r:=w2/(u2+v2+w2);r:=simplify(r);
xP:=(p-q-r)*u+2*p*(v+w);xP:=simplify(xP);
yP:=(q-r-p)*v+2*q*(w+u);yP:=simplify(yP);
zP:=(r-p-q)*w+2*r*(u+v);zP:=simplify(zP);
u3:=3*a^4-(b^2-c^2)^2-2*a^2*(b^2+c^2);
v3:=3*b^4-(c^2-a^2)^2-2*b^2*(c^2+a^2);
w3:=3*c^4-(a^2-b^2)^2-2*c^2*(a^2+b^2);
u:=u3/(u3+v3+w3);v:=v3/(u3+v3+w3);w:=w3/(u3+v3+w3);
u4:=b+c;v4:=c+a;w4:=a+b;
p:=u4/(u4+v4+w4);q:=v4/(u4+v4+w4);r:=w4/(u4+v4+w4);
xQ:=(p-q-r)*u+2*p*(v+w);xQ:=simplify(xQ);
yQ:=(q-r-p)*v+2*q*(w+u);yQ:=simplify(yQ);
zQ:=(r-p-q)*w+2*r*(u+v);zQ:=simplify(zQ);
simplify(xP-xQ);simplify(yP-yQ);simplify(zP-zQ);

```

## 5. EXTENSIONS

In the above investigation we have selected as a starting set of points the set containing the four classical remarkable points: the centroid, incenter, circumcenter and orthocenter. We could extend this set by adding to it a few additional points. Suppose that we form a starting set of points containing the first 12 points of (Kimberling). We use again the operation of the “Discoverer” named the “reflection of a point in a point”. In this case, “Discoverer” produces an extended list of reflection of points which contains 132 remarkable points. Of these 61 are available in (Kimberling), and the rest of 71 remarkable points are not available in (Kimberling). The lists, produced by “Discoverer” are enclosed to this paper. See the folder “Reflections 1-12”. Hence, now we obtain 71 new remarkable points. The “Discoverer” easily could discover new theorems about these new remarkable points.

## 6. PREPARATION OF A SCHOLARLY ESSAY BY A STUDENT

Below we suggest a procedure for “learning through discovery” for the special case of the investigation of the reflections of remarkable points in the geometry of the triangle. We suppose that a student wants to write a scholarly essay (or a bachelor’s thesis, or a master’s thesis, or a scientific paper) by using the help of “Discoverer”. The procedure is as follows:

1. The student chooses a set of remarkable points in the plane of  $\triangle ABC$ .
2. “Discoverer” produces a list of the reflections of these points. Note that the student could calculate these points by hand.

3. “Discoverer” discovers which of the above produced points are not included in (Kimberling). The help of “Discoverer” at this stage is essential.
4. “Discoverer” discovers new theorems about the points which are not available in (Kimberling). The help of “Discoverer” at this stage is essential.
5. The student uses the computer program for dynamic geometry, like C.a.R., in order to investigate the ruler-and-compass constructions of the new points. The student produces macros for the new points and animations for the ruler-and-compass constructions. The student produces also computer graphics for his or her essay.
6. The student uses computer algebra system, like Maple, in order to prepare the proofs of the theorems of the essay. The student calculates also the barycentric coordinates of the new points.
7. The student prepares the scholarly essay. The essay contains as supplementary material the HTML-files, produced by “Discoverer”, the C.a.R. files and graphics, and the Maple files, produced by the student.
8. The new theorems, together with the barycentric coordinates of the new points may be submitted for publication in the Kimberling’s encyclopedia (Kimberling). The student has to submit his or her essay for publication in the Journal of Computer-Generated Mathematics, [www.ddekov.eu/j/](http://www.ddekov.eu/j/), and in the Computer-Generated Encyclopedia of Euclidean Geometry, [eg-enc.webege.com](http://eg-enc.webege.com).

During the work on the scholarly essay, the student will improve his skills to use a discovery system, like “Discoverer”, a system for dynamic geometry, like C.a.R. (or GeoGebra, or Cabri), and a computer algebra system, like Maple (or Mathematica, or Derive). These skills are between the basic skills which the student has to master and improve during his education.

### **Supplementary material**

The file “2015-1\_discovery.zip” contains the files quoted in this paper. The reader may download this file from [http://www.ddekov.eu/papers/2015-1\\_discovery.zip](http://www.ddekov.eu/papers/2015-1_discovery.zip)

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