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Free products with amalgamation of monoids

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Abstract

By using the technique of rewriting, we give a new proof of Bourbaki's theorem on free products with amalgamation of monoids. © 1998 Elsevier Science B.V.

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1. Introduction

Bourbaki [2] has proved the following theorem (we follow the notation of Serre [12, Ch. I, Theorem 1]):

Theorem (Bourbaki [2, Proposition 5, § 7, p. I. 81]). *Let G_i, $i \in I$, be a family of monoids, let A be a submonoid of G_i for all $i \in I$, and let $G_i \cap G_j = A$ for all $i, j \in I$ with $i \neq j$. Let 1 denote the identity element of A, let $u \cdot v$ denote the product of $u, v \in G_i$, and let $*_A G_i$ denote the free product with amalgamation of the G_i. Assume that for every $i \in I$ there exists a subset S_i of G_i containing 1 and such that the mapping $\psi : (a, s_i) \mapsto a \cdot s_i$ from $A \times S_i$ into G_i is a bijection. Then every $g \in *_A G_i$ can be written uniquely in the form $g = as_1 \ldots s_n$ where $a \in A$,*

$$s_1 \in S_{i_1} - \{1\}, \ldots, s_n \in S_{i_n} - \{1\}, \qquad i_m \neq i_{m+1}, \quad 1 \le m \le n-1.$$

It is well known that the main special case of the above Bourbaki's theorem is the theorem of Schreier [11] on free products with amalgamation of groups (see also [12, Ch. I, Theorem 1], or [9], or [5]).

In 1975, Lallement [6] gave a new proof of Bourbaki's theorem.

In 1980, a new approach to free products with amalgamation of groups is due to Evans [4]. Evans' approach is based on the rewriting technique.

In the present paper we give a new proof of Bourbaki's theorem. We use Evans' method [4, § 3] with two changes. We define a new rewriting system R in order to

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prove the confluence of R easily, and further, we use a reduction ordering in order to prove the termination of R.

2. Preliminaries

In this section we review some basic facts about rewriting systems.

We refer the reader to [3] or [1] for a survey on rewriting systems.

Let X be a set and let X^* be the free monoid on X, the empty word of which is denoted by λ. A *rewriting system* (or a *string-rewriting system*) on X is a subset $R \subseteq X^* \times X^*$. An element $(l, r) \in R$, also written $l \to r$, is called a *rule* of R. The *single-step reduction relation* on X^* induced by R, which by abuse of notation will also be denoted by $\to$, is defined as follows:

$$u \to v \quad \text{iff} \quad \exists x, y \in X^* \; \exists (l \to r) \in R\colon u = xly \quad \text{and} \quad v = xry.$$

Its reflexive transitive closure $\xrightarrow{*}$ is the *reduction relation* induced by R.

A rewriting system R on X is called

- *terminating* if for any word $x \in X^*$ there is no infinite chain of single-step reductions $x \to x_1 \to x_2 \to \cdots$;
- *confluent* if for any reductions $x \xrightarrow{*} y$ and $x \xrightarrow{*} z$ there exists a $w \in X^*$ such that $y \xrightarrow{*} w$ and $z \xrightarrow{*} w$;
- *complete* if it is both terminating and confluent.

Henceforth we call an irreflexive and transitive binary relation an *ordering*.

Let $>$ be an ordering on X^*. This ordering is called

- *monotonic* if it is compatible with the operation of concatenation, i.e., for all $u, v, x, y \in X^*$, if $u > v$, then also $xuy > xvy$;
- *well-founded* if for any word $x \in X^*$ there does not exist an infinite descending chain $x > x_1 > x_2 > \cdots$;
- *reduction* if it is both monotonic and well-founded.

Theorem A (Lankford [7]). *A rewriting system R on X is terminating if and only if there exists a reduction ordering $>$ on X^* such that $l > r$ for each rule $(l \to r) \in R$.*

A function $\varphi : X \to \mathbb{N}$ satisfying $\varphi(a) > 0$ for all $a \in X$ is called a *weight-function*. It can uniquely be extended to a homomorphism from X^* into $\mathbb{N}$, which by abuse of notation will also be denoted by φ.

Let $u, v \in X^*$, let φ be a weight-function, let $\rhd$ be a well-founded ordering on X, called a *precedence* on X, let $>_{\text{Lex-L}}$ be the lexicographic ordering from the left on X^* induced by the precedence $\rhd$, and let $>$ denote the usual ordering on $\mathbb{N}$. Define the weight-plus-lexicographic ordering from the left (WLO-L, for short) as follows:

$$u >_{\text{WLO-L}} v \quad \text{iff} \quad \text{either} \quad \varphi(u) > \varphi(v) \quad \text{or} \quad \varphi(u) = \varphi(v) \quad \text{and} \quad u >_{\text{Lex-L}} v.$$

It is easy to verify that the WLO-L is a reduction ordering.

Let $(uv \to s) \in R$, $(vw \to t) \in R$ and u, v, w are nonempty words. Then the word uvw is called an *overlap ambiguity* of R. Let $(v \to s) \in R$, $(uvw \to t) \in R$ and let $u = \lambda$ and $w = \lambda$ imply $s \neq t$. Then the word uvw is called an *inclusion ambiguity* of R. The pair of words (sw, ut) or (usw, t), respectively, is called a *critical pair* of R. A critical pair (p, q) of R is *resolved* if there is a word $z \in X^*$ such that $p \xrightarrow{*} z$ and $q \xrightarrow{*} z$.

Theorem B (Newman [10]). *Let R be a terminating rewriting system. Then R is confluent if and only if all critical pairs of R are resolved.*

Given a semigroup S, a rewriting system R on X is called a *rewriting system for S if*

$$\mathrm{sgp}(X;\, l = r \text{ where } (l \to r) \in R)$$

is a presentation for S. A word $u \in X^*$ is called *R-irreducible* if there is no single-step reduction $u \to v$ for some $v \in X^*$.

Theorem C (Newman [10]). *If R is a complete rewriting system for a semigroup S, then there is exactly one R-irreducible word representing each element of S.*

3. An example

We give an example of an ordering which is not a reduction ordering.

Let $X = \{a, b, c\}$. Define a weight-function by $\varphi(a) = 1$, $\varphi(b) = 3$, $\varphi(c) = 5$. For $u = x_1 x_2 \dots x_n \in X^*$, we set

$$\#(ba, u) = \mathrm{card}(\{(i, j) : i < j,\ x_i = b,\ x_j = a\}).$$

E.g., $\#(ba, bca) = 1$ and $\#(ba, bbaa) = 4$. We use the notation $\#(v, u)$, where $v, u \in X^*$, from [8].

Define the ordering $>_E$ on X^* as follows:

$$u >_E v \quad \text{iff} \quad \text{either} \quad \varphi(u) > \varphi(v) \quad \text{or} \quad \varphi(u) = \varphi(v) \quad \text{and} \quad \#(ba, u) > \#(ba, v).$$

The ordering $>_E$ is not monotonic. Indeed, let $u = cba$ and $v = bbb$. Then $u >_E v$, since $\varphi(u) = \varphi(v) = 9$ and $\#(ba, u) = 1 > 0 = \#(ba, v)$. But $ua = cbaa \not>_E va = bbba$, since $\varphi(ua) = \varphi(va) = 10$ and $\#(ba, ua) = 2 < 3 = \#(ba, va)$. Since the ordering $>_E$ is not monotonic, it is not a reduction ordering (cf. [4, p. 98]).

4. Proof of the theorem

Let $G = \bigcup \{G_i : i \in I\}$. Then

$$P = *_A G_i = \mathrm{sgp}(G;\, uv = u \cdot v \text{ where } u, v \in G_i,\ i \in I).$$

Since ψ is a bijection, the subsets A, $S_i - \{1\}$ and $G_i - (A \cup S_i)$, $i \in I$, form a partition of G. Denote $\bar{S}_i = S_i - \{1\}$ and $\bar{G}_i = G_i - (A \cup S_i)$, $i \in I$. Define the rewriting systems

$$R_A = \{a_1 a_2 \to a: a, a_1, a_2 \in A \text{ and } a_1 \cdot a_2 = a\},$$

$$\begin{aligned} R_i = \{ & x_i \to a s_i: a \in A,\ s_i \in \bar{S}_i,\ x_i \in \bar{G}_i \text{ and } a \cdot s_i = x_i; \\ & s_{i1} s_{i2} \to a: a \in A,\ s_{i1}, s_{i2} \in \bar{S}_i \text{ and } s_{i1} \cdot s_{i2} = a; \\ & s_{i1} s_{i2} \to a s_i: a \in A,\ s_i, s_{i1}, s_{i2} \in \bar{S}_i \text{ and } s_{i1} \cdot s_{i2} = a \cdot s_i; \\ & s_i a_1 \to a_2: a_1, a_2 \in A,\ s_i \in \bar{S}_i \text{ and } s_i \cdot a_1 = a_2; \\ & s_{i1} a_1 \to a_2 s_{i2}: a_1, a_2 \in A,\ s_{i1}, s_{i2} \in \bar{S}_i \text{ and } s_{i1} \cdot a_1 = a_2 \cdot s_{i2}\}, \end{aligned}$$

$$R = R_A \cup \left(\bigcup \{R_i: i \in I\} \right).$$

It is routine to verify that R is a rewriting system for the semigroup P.

The rewriting system R is terminating. Indeed, let $>_{\mathrm{WLO\text{-}L}}$ denote the weight-plus-lexicographic ordering from the left on G^* defined by the weight-function $\varphi(a) = 1$, for all $a \in A$, $\varphi(s_i) = 2$, for all $s_i \in \bar{S}_i$, $\varphi(x_i) = 4$, for all $x_i \in \bar{G}_i$ and the precedence $s_i \rhd a$ for all $a \in A$, $s_i \in \bar{S}_i$. Then $l >_{\mathrm{WLO\text{-}L}} r$ for all $(l \to r) \in R$. Since $>_{\mathrm{WLO\text{-}L}}$ is a reduction ordering, by Theorem A, R is terminating.

The rewriting system R is confluent. Indeed, the only overlap ambiguities are of the form uvw, where u, v, w belong to the same monoid. The rewriting system R does not have any inclusion ambiguities. Clearly, all critical pairs of R are resolved. Hence, by Theorem B, R is confluent.

R is complete, since it is terminating and confluent.

The R-irreducible words (distinct from λ) are of the form

$$a s_1 \ldots s_n$$

where $a \in A$, $s_1 \in \bar{S}_{i_1}, \ldots, s_n \in \bar{S}_{i_n}$, $i_m \neq i_{m+1}$, $1 \le m \le n-1$.

Since R is a complete rewriting system for P, by Theorem C, there is exactly one R-irreducible word representing each element of P. This completes the proof of the Theorem.

Note: If A and G_i, $i \in I$, are groups, then the definition of the R_i can be slightly simplified, since the subsets $\{s_i a_1 \to a_2: a_1, a_2 \in A,\ s_i \in \bar{S}_i \text{ and } s_i \cdot a_1 = a_2\}$ are empty.

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