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RESEARCH ARTICLE

Free Products with Amalgamation of Semigroups

Deko V. Dekov

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1. Introduction

In 1927 Schreier [26] showed that a group amalgam is always embeddable in a group. A new approach to the embeddability of group amalgams, based on partial groupoids, is due to R. Baer [1]-[3]. R. Baer's approach yields a new proof of Schreier's theorem.

Let $\{S_i : i \in I\}$ be a family of semigroups, let U be a subsemigroup of S_i for all $i \in I$, and let $S_i \cap S_j = U$ for all $i, j \in I$ with $i \neq j$. The semigroup amalgam $\mathcal{A} = [\{S_i : i \in I\}; U]$ determines a partial groupoid $\mathcal{G}_{\mathcal{A}}$, and the amalgam is said to be embeddable in a semigroup if $\mathcal{G}_{\mathcal{A}}$ is embeddable in a semigroup ([4], Section 9.4).

In 1957 N. Kimura ([17]; see also [4]) showed that a semigroup amalgam cannot always be embedded in a semigroup. Semigroup amalgams were first extensively studied by J. M. Howie ([10]–[15]; see also [4], [16]). In 1962 J. M. Howie [10] showed that if U is an almost unitary subsemigroup of S_i for all $i \in I$, then the amalgam is embeddable. In particular, if U is a unitary subsemigroup of S_i for all $i \in I$, then the amalgam is embeddable. New proofs of J. M. Howie's result concerning amalgamation over unitary subsemigroups are due to G. B. Preston [23], T. E. Hall [9], G. Lallement [18], [19] and D. Dekov [5].

In 1975 T. E. Hall [8] showed that Schreier's theorem extends to the class of inverse semigroups.

The purpose of the present paper is to extend R. Baer's method to the case of semigroup amalgams.

If $\mathcal{A} = [\{S_i : i \in I\}; U]$ is a group amalgam, then in its free product amalgamating U equivalent reduced words have the same length. In the present paper we describe semigroup amalgams having the above property, i.e., semigroup amalgams embeddable in a semigroup and such that in the free products amalgamating the common subsemigroups equivalent reduced words have the same length. A particular case is the following result ([5], Theorem 2): Let $\mathcal{A}$ be a semigroup amalgam. If U is unitary in S_i for all $i \in I$, then $\mathcal{A}$ is embeddable and in its free product amalgamating U equivalent reduced words have the same length.

Let $\{S_i : i \in I\}$ be a family of groups (or semigroups). Then in the free product of the S_i, $i \in I$ equivalent reduced words are equal. A description of semigroup amalgams, having the above property, i.e., semigroup amalgams such that in the free products amalgamating the common subsemigroups equivalent reduced words are equal, is due to E. S. Lyapin [20]–[22] (for the case $|I| = 2$). In the present paper we give a new proof of E. S. Lyapin's theorem. J.-C. Spehner [28], using R. Baer's approach has proved the following particular case: Let $\mathcal{A}$ be a semigroup amalgam. If $\mathcal{G}_{\mathcal{A}}$ is associative, and if U is an ideal of S_i for all

$i \in I$, then $\mathcal{A}$ is embeddable and in its free product amalgamating U equivalent reduced words are equal. A particular case is P. A. Grillet and M. Petrich's result [7]: Let $\mathcal{A}$ be a semigroup amalgam. If $\mathcal{G}_{\mathcal{A}}$ is associative and if U is an ideal of S_i for all $i \in I$, then the amalgam is embeddable.

2. Preliminaries

In this section we review some of R. Baer's results on partial semigroups.

A *partial groupoid* is a triple $G = (G, D, \mu)$ where G is a non-empty set, $D \subseteq G \times G$ and $\mu : D \to G$ is a mapping. We use these notations ([29]):

$$xy = \mu(x, y)$$
$$(x, y)_D \quad \text{iff} \quad (x, y) \in D\ .$$

Let $G = (G, D, \mu)$ be a partial groupoid and let $F(G)$ be the free semigroup on the set G. Let $X = (x_1, \ldots, x_n) \in F(G)$. Then X is called a *word* of *length* n. A word $(x_1, \ldots, x_n)$ is said to be *reduced*, if for $i = 1, 2, \ldots, n-1$, $(x_i, x_{i+1}) \notin D$. We will use the following notation ([24, [25]):

$$(x_1, \ldots, x_n)_D \quad \text{iff} \quad (x_i, x_{i+1})_D \quad \text{for} \quad i = 1, 2, \ldots, n-1\ .$$

Let G be a partial groupoid and let θ_0 be the relation on $F(G)$ defined as follows: $(xy, (x, y)) \in \theta_0$ iff $(x, y)_D$. Let θ be the congruence on $F(G)$ generated by θ_0. Then $U(G) = F(G)/\theta$ is the *universal semigroup* of G. The words X and Y are *equivalent*, if $X \sim Y \pmod{\theta}$.

Definition 1. (J.-C. Spehner [27]) A partial groupoid G is said to be a *partial semigroup* if it is embeddable in a semigroup.

Theorem 1. (R. Baer [1]) *A partial groupoid G is a partial semigroup if and only if $x \sim y \pmod{\theta}$ implies $x = y$, for all $x, y \in G$.*

Definition 2. A partial semigroup G is said to be an *independent partial semigroup* if it satisfies the following property:

(U) Equivalent reduced words are equal.

Definition 3. A partial semigroup G is said to be a *presemigroup* if it satisfies the following property:

(L) Equivalent reduced words have the same length.

We introduce the following conditions on a partial groupoid:

(A) For all $x, y, z \in G$, if $(x, y, z)_D$, $(x, yz)_D$ and $(xy, z)_D$, then $x(yz) = (xy)z$.

(B) For all $x, y, z \in G$, if $(x, y, z)_D$, then: $(x, yz)_D$ if and only if $(xy, z)_D$.

(C) For all $x, y, z \in G$, if $(x, y, z)_D$, then $(xy, z) \notin D$ implies $xy = x$, and $(x, yz) \notin D$ implies $yz = z$.

($D.1$) For all $w, x, y, z \in G$, if $(w, x, y, z)_D$, then either $(w, xy)_D$ or $(xy, z)_D$.

($D.2$) For all $w, x, y, z \in G$, if $(x, y, z)_D$ and $(w, xy)_D$, then either $(w, x)_D$ or $(xy, z)_D$.

($D.3$) For all $w, x, y, z \in G$, if $(w, x, y)_D$ and $(xy, z)_D$, then either $(w, xy)_D$ or $(y, z)_D$.

Theorem 2. (R. Baer [3], §3, Theorem 1) *A partial groupoid G is an independent partial semigroup if and only if it satisfies axioms (A), (B) and (C).*

Theorem 3. (R. Baer, [3], §1, Theorem 1) *A partial groupoid G is a presemigroup if and only if it satisfies axioms (A), (B), $(D.1)$, $(D.2)$ and $(D.3)$.*

3. Quasi-unitary subsemigroups

In order to give sufficient conditions under which a semigroup amalgam can be embedded in a semigroup, J. M. Howie ([10]; see also [4], [16]) has generalized the concept of a unitary subsemigroup to the concept of an almost unitary subsemigroup.

In order to describe semigroup amalgams which are embeddable and have the property that in the free products amalgamating the common subsemigroups, equivalent reduced words have the same length, we will generalize the concepts of a unitary subsemigroup, an ideal and a separable unit ideal subsemigroup, and introduce the concept of a quasi-unitary subsemigroup.

These two generalizations are independent, neither implies the other.

Definition 4. (P. Dubreil [6]) A subsemigroup U of a semigroup S is said to be *unitary* in S if, for all $u \in U$, $s \in S$,

(i) $us \in U \Longrightarrow s \in U$.

(ii) $su \in U \Longrightarrow s \in U$.

Let S be a semigroup and let U be a subsemigroup of S. We define

$$\begin{aligned} L(S) &= \{u \in U : \forall\, s \in S\backslash U, su \in U\} , \\ L'(S) &= \{u \in U : \forall\, s \in S\backslash U, su = s\} , \\ L''(S) &= \{u \in U : \forall\, s \in S\backslash U, su \in S\backslash U\} , \\ R(S) &= \{u \in U : \forall\, s \in S\backslash U, us \in U\} , \\ R'(S) &= \{u \in U : \forall\, s \in S\backslash U, us = s\} , \\ R''(S) &= \{u \in U : \forall\, s \in S\backslash U, us \in S\backslash U\} . \end{aligned}$$

These sets are empty or subsemigroups of U. Moreover, $L(S)$ $[R(S)]$ is empty or a left [right] ideal of S. Clearly,

$$L(S) \cap L'(S) = \emptyset,\ L(S) \cap L''(S) = \emptyset,\ R(S) \cap R'(S) = \emptyset,\ R(S) \cap R''(S) = \emptyset,$$
$$L'(S) \subseteq L''(S),\ R'(S) \subseteq R''(S) .$$

Definition 5. (E. S. Lyapin [21]) A subsemigroup U of a semigroup S is said to be a *separable unit ideal* in S if $U = L(S) \cup L'(S) = R(S) \cup R'(S)$.

Definition 6. A subsemigroup U of a semigroup S is said to be *quasi-unitary* in S if $U = L(S) \cup L''(S) = R(S) \cup R''(S)$.

Let U be a quasi-unitary subsemigroup of S. Clearly,

U is unitary in S if and only if $L(S) = R(S) = \emptyset$;

U is an ideal in S if and only if $L''(S) = R''(S) = \emptyset$;

U is a separable unit ideal in S if and only if $L'(S) = L''(S)$ and $R'(S) = R''(S)$.

In the next section we will use the following

Lemma 1. *Let U be a proper subsemigroup of a semigroup S. Let $s \in S\backslash U$ and $u \in U$. The following conditions are equivalent:*

(1) $\exists\ s(su \in U) \Longrightarrow \forall\ s(su \in U)$.

(2) $\exists\ s(su \in S\backslash U) \Longrightarrow \forall\ s(su \in S\backslash U)$.

(3) $U = L(S) \cup L''(S)$.

Likewise the following dual conditions are equivalent:

(1′) $\exists\ s(us \in U) \Longrightarrow \forall\ s(us \in U)$.

(2′) $\exists\ s(us \in S\backslash U) \Longrightarrow \forall\ s(us \in S\backslash U)$.

(3′) $U = R(S) \cup R''(S)$.

Proof. (1)$\Longrightarrow$(2). Let $\exists\ s_0(s_0u \in S\backslash U)$. Suppose that $\exists\ s_1(s_1u \in U)$. Then by (1), $s_0u \in U$, a contradiction.

Similarly, (2)$\Longrightarrow$(1).

(1) and (2)$\Longrightarrow$(3). If the antecedent of (1) holds, then $u \in L(S)$. If the antecedent of (2) holds, then $u \in L''(S)$. Since U is a proper subsemigroup of S, either the antecedent of (1) holds or the antecedent of (2) holds. Hence $u \in L(S)$ or $u \in L''(S)$, i.e., $U = L(S) \cup L''(S)$.

(3)$\Longrightarrow$(1). Suppose that $\exists\ s(su \in U)$. Then $u \notin L''(S)$. By (3), $u \in L(S)$, i.e. $\forall\ s(su \in U)$.

Similarly, conditions (1′), (2′) and (3′) are equivalent. Lemma 1 is proved.

Using Lemma 1 we see that a proper subsemigroup U of a semigroup S is quasi-unitary in S if and only if both (1) and (1′) hold, or equivalently, both (2) and (2′) hold.

4. Semigroup amalgams whose partial groupoids are presemigroups

A *semigroup amalgam* consists of a semigroup U together with a family $\{S_i : i \in I\}$ of semigroups and a family $\{\varphi_i : i \in I\}$ of monomorphisms $\varphi_i : U \to S_i$. We denote the semigroup amalgam by

$$\mathcal{A} = [\{S_i : i \in I\}; U; \{\varphi_i : i \in I\}] \ .$$

The semigroup amalgam determines a partial groupoid $\mathcal{G}_\mathcal{A}$ and the amalgam is said to be *embeddable* in a semigroup if $\mathcal{G}_\mathcal{A}$ is a partial semigroup ([4], Section 9.4). As in [4], write $U_i = U\varphi_i$, $S_i' = (S_i\backslash U_i) \cup U$ and $G = \cup\{S_i' : i \in I\}$. We denote $\mathcal{G}_\mathcal{A} = (G, D)$ and $L_i = L(S_i')$, $L_i' = L'(S_i')$ and so on. We suppose $S_i' \cap S_j' = U$ for all $i, j \in I$ with $i \neq j$. Without loss of generality we assume that U is a proper subsemigroup of S_i' for all $i \in I$.

Lemma 2. ([5]) *For any semigroup amalgam $\mathcal{A} = [\{S_i : i \in I\}; U; \{\varphi_i : i \in I\}]$, the partial groupoid $\mathcal{G}_\mathcal{A}$ satisfies axioms* (D.1), (D.2) *and* (D.3).

The following theorem gives a description of semigroup amalgams which are embeddable and have the property that in the free products amalgamating the common subsemigroups, equivalent reduced words have the same length.

Theorem 4. *Let $\mathcal{A} = [\{S_i : i \in I\}; U; \{\varphi_i : i \in I\}]$ be a semigroup amalgam. The partial groupoid $\mathcal{G}_\mathcal{A}$ is a presemigroup if and only if it satisfies the following conditions:*

(i) *U is a quasi-unitary subsemigroup of S_i' for all $i \in I$.*

(ii) *$L_i = R_j$ for all $i, j \in I$ with $i \neq j$.*

(iii) $s_i(us_j) = (s_iu)s_j$ *for all* $s_i \in S_i'\backslash U$, $s_j \in S_j'\backslash U$, $u \in L_i$, *and all* $i, j \in I$ *with* $i \neq j$.

Proof. By Theorem 3, $\mathcal{G}_{\mathcal{A}}$ is a presemigroup if and only if it satisfies axioms (A), (B), $(D.1)$, $(D.2)$ and $(D.3)$. By Lemma 2, $\mathcal{G}_{\mathcal{A}}$ satisfies $(D.1)$, $(D.2)$ and $(D.3)$. We will show that $\mathcal{G}_{\mathcal{A}}$ satisifes (A) and (B) if and only if it satisfies the conditions (i), (ii) and (iii).

Let $x, y, z \in G$ and $(x, y, z)_D$. Then x, y and z belong to the same semigroup, except for the case when $x \in S_i'\backslash U$, $y \in U$ and $z \in S_j'\backslash U$ for some $i, j \in I$ with $i \neq j$. So that it is enough to consider only the above case.

Let $i, j \in I$ with $i \neq j$. We will denote by s_i an arbitrary element in $S_i'\backslash U$, by u an arbitrary element in U, and by s_j an arbitrary element in $S_j'\backslash U$. Clearly, $(s_iu, s_j)_D \iff s_iu \in U$, and $(s_i, us_j)_D \iff us_j \in U$. The partial groupoid $\mathcal{G}_{\mathcal{A}}$ satisfies axiom (B) if and only if it satisfies the following conditions:

$$\exists\ s_i\,[(s_iu, s_j)_D] \Longrightarrow \forall\ s_j\,[(s_i, us_j)_D]\ ,$$
$$\exists\ s_j\,[(s_i, us_j)_D] \Longrightarrow \forall\ s_i\,[(s_iu, s_j)_D]\ ,$$

or equivalently, the conditions:

$$\exists\ s_i(s_iu \in U) \Longrightarrow \forall\ s_j(us_j \in U)\ ,$$
$$\exists\ s_j(us_j \in U) \Longrightarrow \forall\ s_i(s_iu \in U)\ ,$$

or equivalently, the conditions:

$$(a)\qquad \exists\ s_i(s_iu \in U) \Longrightarrow \forall\ s_i(s_iu \in U)\ ,$$
$$(b)\qquad \exists\ s_j(us_j \in U) \Longrightarrow \forall\ s_j(us_j \in U)\ ,$$
$$(c)\qquad \forall\ s_i(s_iu \in U) \Longleftrightarrow \forall\ s_j(us_j \in U)\ .$$

Changing the roles of i and j, we see that (b) is equivalent to the condition

$$(b')\qquad \exists\ s_i(us_i \in U) \Longrightarrow \forall\ s_i(us_i \in U)\ .$$

By Lemma 1, $\mathcal{G}_{\mathcal{A}}$ satisfies (a) and (b') if and only if it satisfies (i). Condition (c) is equivalent to the condition

$$u \in L_i \Longleftrightarrow u \in R_j\ ,$$

i.e.,

$$L_i = R_j\ .$$

We have proved that $\mathcal{G}_{\mathcal{A}}$ satisfies (B) if and only if it satisfies (i) and (ii). Next we prove that if $\mathcal{G}_{\mathcal{A}}$ satisfies (ii), then $\mathcal{G}_{\mathcal{A}}$ satisfies (A) if and only if it satisfies (iii). We have $(s_iu, s_j)_D \iff u \in L_i$, and $(s_i, us_j)_D \iff u \in R_j$. By (ii), $L_i = R_j$, so that both $(s_iu, s_j)_D$ and $(s_i, us_j)_D$ hold if and only if $u \in L_i$. Hence $\mathcal{G}_{\mathcal{A}}$ satisfies (A) if and only if it satisfies (iii). This completes the proof of Theorem 4.

Remark 1. If $\mathcal{G}_{\mathcal{A}}$ is a presemigroup, then the case $L_i \neq R_i$ is possible for some $i \in I$ if and only if $|I| = 2$. Indeed, suppose that $|I| \geq 3$. Then there exist $i, j, k \in I$ with $i \neq j$, $j \neq k$, $i \neq k$. Condition (ii) implies

$$L_i = R_j, \quad L_k = R_j, \quad L_k = R_i\ .$$

Hence $L_i = R_i$.

Remark 2. If $\mathcal{G}_{\mathcal{A}}$ satisifies (i), then, obviously, (ii) is equivalent to the condition

(ii′) $$L_i'' = R_j'' \quad \text{for all} \quad i,j \in I \quad \text{with} \quad i \neq j \; .$$

Setting in Theorem 4 $L_i = R_i = \emptyset$ for all $i \in I$, we obtain

Corollary 1. ([5]) *Let $\mathcal{A} = [\{S_i : i \in I\}; U; \{\varphi_i : i \in I\}]$ be a semigroup amalgam. If U is a unitary subsemigroup of S_i for all $i \in I$, then $\mathcal{G}_{\mathcal{A}}$ is a presemigroup.*

5. Semigroup amalgams whose partial groupoids are independent partial semigroups

A description of semigroup amalgams such that in the free products amalgamating the common subsemigroups, equivalent reduced words are equal, is due to E. S. Lyapin.

Theorem 5 below was proved by E. S. Lyapin [20]–[22] (for the case $|I| = 2$). Here we will give a new proof of E. S. Lyapin's theorem. Theorem 5 can be proved in the same way as Theorem 4. Here we will give a proof of Theorem 5 as a corollary to Theorem 4.

Theorem 5. *Let $\mathcal{A} = [\{S_i : i \in I\}; U; \{\varphi_i : i \in I\}]$ be a semigroup amalgam. The partial groupoid $\mathcal{G}_{\mathcal{A}}$ is an independent partial semigroup if and only if it satisfies the following conditions:*

(i) *U is a separable unit ideal subsemigroup of S_i for all $i \in I$.*
(ii) *$L_i = R_j$ for all $i,j \in I$ with $i \neq j$.*
(iii) *$s_i(us_j) = (s_iu)s_j$ for all $s_i \in S_i'\backslash U, s_j \in S_j'\backslash U$, $u \in L_i$, and all $i,j \in I$ with $i \neq j$.*

Proof. By Theorem 2, $\mathcal{G}_{\mathcal{A}}$ is an independent partial semigroup if and only if it satisfies axioms (A), (B) and (C). Let $\mathcal{G}_{\mathcal{A}}$ be a presemigroup. Then $\mathcal{G}_{\mathcal{A}}$ is an independent partial semigroup if and only if it satisfies axiom (C). We will prove that $\mathcal{G}_{\mathcal{A}}$ satisfies axiom (C) if and only if it satisfies the following condition:

(i′) $$L_i' = L_i'' \quad \text{and} \quad R_i' = R_i'' \quad \text{for all} \quad i \in I \; .$$

As in the proof of Theorem 4, suppose that $i,j \in I$, $i \neq j$, $s_i \in S_i'\backslash U$, $s_j \in S_j'\backslash U$, $u \in U$. We have $(s_i,u,s_j)_D$. Clearly, $(s_iu,s_j) \notin D \iff s_iu \in S_i'\backslash U \iff u \in L_i''$, and $(s_i,us_j) \notin D \iff us_j \in S_j'\backslash U \iff u \in R_j''$. The partial groupoid $\mathcal{G}_{\mathcal{A}}$ satisfies axiom (C) if and only if it satisfies the following conditions:

$$(s_iu,s_j) \notin D \Longrightarrow s_iu = s_i \; ,$$
$$(s_i,us_j) \notin D \Longrightarrow us_j = s_j \; ,$$

or equivalently, the conditions:

$$s_iu \in S_i'\backslash U \Longrightarrow s_iu = s_i \; ,$$
$$us_j \in S_j'\backslash U \Longrightarrow us_j = s_j \; ,$$

or equivalently, the conditions:

$$u \in L_i'' \Longrightarrow u \in L_i' \ ,$$
$$u \in R_j'' \Longrightarrow u \in R_j' \ ,$$

Changing the roles of i and j, we see that the above conditions are equivalent to the conditions: $L_i'' \subseteq L_i'$ and $R_i'' \subseteq R_i'$, or equivalently, to (i′). Since U is a quasi-unitary subsemigroup of S_i, $i \in I$, (i′) is equivalent to (i). (See Section 3). Hence $\mathcal{G}_{\mathcal{A}}$ satisfies axiom (C) if and only if it satisfies (i). We see that Theorem 5 is a corollary to Theorem 4. This completes the proof of Theorem 5.

Setting $L_i' = L_i''$ and $R_i' = R_i''$ in Remarks to Theorem 4, we obtain the corresponding remarks to Theorem 5.

Corollary 2 below was proved by J.-C. Spehner [28] by using R. Baer's results on independent partial semigroups. Setting in Theorem 5, $L_i' = R_i' = \emptyset$, for all $i \in I$, we obtain:

Corollary 2. (J.-C. Spehner [28]) *Let* $\mathcal{A} = [\{S_i : i \in I\}; U; \{\varphi_i : i \in I\}]$ *be a semigroup amalgam. If* $\mathcal{G}_{\mathcal{A}}$ *satisfies* (A) *and if* U *is an ideal of* S_i' *for all* $i \in I$, *then* $\mathcal{G}_{\mathcal{A}}$ *is an independent partial semigroup.*

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Georgi Kolev 81
6000 Stara Zagora
Bulgaria

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