98.33 Computer-generated mathematics: points on the Kiepert hyperbola

In 1869 Ludwig Kiepert [1] introduced a hyperbola, now known as the Kiepert hyperbola. During the years a number of remarkable points of the triangle have been discovered to lie on the Kiepert hyperbola. In 1994 Eddy and Fritsch discovered that the Spieker centre and the third Brocard point lie on the Kiepert hyperbola ([2, Theorems 3 and 4]). Eric Weisstein [3] has presented a list of 44 points which lie on the Kiepert hyperbola.

We remind the reader of a few definitions. Given $\triangle ABC$ and a point P, the *complement of the point P* is its image under a homothety with centre the centroid G and scale factor $-\frac{1}{2}$. If three lines through the vertices of $\triangle ABC$ are concurrent in a point P, then their reflections in the angle bisectors are concurrent in a point called the *isogonal conjugate* of P. Given points P and Q, let $P_A P_B P_C$ be the anticevian triangle* of a point P and let DEF be the triangle inscribed in $\triangle ABC$ obtained by $D = P_A Q \cap BC$ with similar expressions for E and F. Then the lines AD, BE and CF concur in a point called the *Ceva product of P and Q*.

In the theorem below we use barycentric coordinates with respect to the reference triangle ABC in the Euclidean plane with points at infinity. We remind the reader of the formulas for calculation in barycentric coordinates of the above-defined notions. We denote by a, b, c the side-lengths of triangle ABC, $a = BC$, $b = CA$ and $c = AB$. Given a point $P = (u, v, w)$, the point $(v + w, w + u, u + v)$ is the complement of P, and the point $\left(\frac{a^2}{u}, \frac{b^2}{v}, \frac{c^2}{w}\right)$ is the isogonal conjugate of P. Given points $P = (u, v, w)$ and $Q = (d, e, f)$, the Ceva product of P and Q is the point

$$P * Q = \big((uf + wd)(ue + vd), (vd + ue)(vf + we), (we + vf)(wd + uf)\big).$$

The barycentric equation of the Kiepert hyperbola is as follows [4, pp. 376-378]:

$$\left(b^2 - c^2\right)yz + \left(c^2 - a^2\right)zx + \left(a^2 - b^2\right)xy = 0. \qquad (1)$$

Theorem: Let P and Q, neither lying on a side of $\triangle ABC$. If P and Q are isogonal conjugates with respect to $\triangle ABC$, then the Ceva product of their complements lies on the Kiepert hyperbola.

Proof: Taking $P = (u, v, w)$, and denoting its complement by $\tilde{P}$, we have

$$\tilde{P} = (v + w, w + u, u + v), Q = \left(\frac{a^2}{u}, \frac{b^2}{v}, \frac{c^2}{w}\right), \tilde{Q} = \left(\frac{b^2}{v} + \frac{c^2}{w}, \frac{c^2}{w} + \frac{a^2}{u}, \frac{a^2}{u} + \frac{b^2}{v}\right)$$

and so if $\tilde{P} * \tilde{Q} = (p, q, r)$ we have

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* P and P_A, P and P_B, P and P_C are harmonic conjugates [4, p. 59] with respect to A and K, B and L, C and M respectively, where $K = AP \cap BC$, $L = BP \cap CA$ and $M = CP \cap AB$.

$$p = \left[(v+w)\left(\frac{a^2}{u}+\frac{b^2}{v}\right)+(u+v)\left(\frac{b^2}{v}+\frac{c^2}{w}\right)\right]\left[(v+w)\left(\frac{c^2}{w}+\frac{a^2}{u}\right)+(w+u)\left(\frac{b^2}{v}+\frac{c^2}{w}\right)\right]$$

with similar expressions for q, r. Then it is enough to check that

$$\left(b^2 - c^2\right)qr + \left(c^2 - a^2\right)rp + \left(a^2 - b^2\right)pq = 0.$$

If we want to avoid calculations by hand, we may use a computer algebra system, such as *Maple*, *Mathematica* or *Derive*.

The table below gives a few special cases of the Theorem.

	P	$\tilde{P} * \tilde{Q}$
1	Incentre	Spieker Centre
2	Centroid	Isotomic Conjugate of the Complement of the Complement of the Symmedian Point
3	Circumcentre	Isogonal Conjugate of the Centre of the Taylor Circle

Note that the Ceva products in the last two rows in the table are not available in [5] and hence we may expect that they are new remarkable points which lie on the Kiepert hyperbola.

The theorem of this paper was discovered by the computer program 'Discoverer', created by the authors. The proofs of 'Discoverer' are non-standard. In this paper we gave a proof which is rewritten in a traditional manner. In [6] the authors present a list of more than 2000 remarkable points which lie on the Kiepert hyperbola. The list was partially produced by using the theorem of this paper. The complete list was produced by the computer program 'Discoverer'.

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References

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