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COMPUTER-DISCOVERED MATHEMATICS: CEVIAN CORNER PRODUCTS

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Abstract. By using the computer program "Discoverer", we give theorems about cevian corner products.
Keywords: cevian corner product, triangle geometry, remarkable point, computer-discovered mathematics, euclidean geometry, eiscoverer.

1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See (Grozdev & Dekov, 2014a,b, 2015a,b). In this paper, by using the "Discoverer", we investigate the cevian corner products. The paper contains more than 1000 theorems about cevian corner products. We expect that the majority of these theorems are new theorems, discovered by a computer.

Given triangle *ABC*, the side lengths are denoted by $a = BC$, $b = CA$ and $c = BA$. The labeling of triangle centers follows (Kimberling). Hence, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, etc. We refer the reader to (Kimberling, Glossary) for the definition of a triangle center.

In 2003 Eric Danneels has published the following result (See also (Castellsaguer, Point X(81), result 602), (Kimberling, X(81))):

Theorem 1 {Danneels, result 3}. *Let PaPbPc be the incentral triangle of triangle ABC, and Ka, Kb, Kc be the symmedian points of triangles APcPb, BPaPc and CPbPa, respectively. Then the lines AKa, BKb and CKc concur in the Isogonal Conjugate of the Grinberg Point.*

Figure 1 illustrates theorem 1. In Fig.1, *I* is the Incenter, *PaPbPc* is the incentral triangle, *Ka*, *Kb* and *Kc* are the symmedian points of triangles *APcPb*, *BPaPc* and *CPbPa*, respectively. The lines *AKa*, *BKb* and *CKc* concur in point *Q*, the isogonal conjugate of the Grinberg point.

Let P and Q be finite triangle centers. Let $PaPbPc$ be the cevian triangle of P. We say that $\Delta APcPb$, $\Delta BPaPc$ and $\Delta CPbPa$ are the *cevian corner triangles of P*. Denote by Ka the Q-triangle center of $\Delta APcPb$, by Kb the Q-triangle center of $\Delta BPaPc$, and by Kc is the Q-triangle center of $\Delta CPbPa$. If the lines AKa, BKb and CKc concur in a point, we say that the *cevian corner product of P and Q exists*, and we call the point of concurrence of the lines the *cevian corner product of P and Q*. Hence, we can reformulate theorem 1 as follows:

Theorem 1a. *The Cevian Corner Product of the Incenter and the Symmedian Point exists, and it is the Isogonal Conjugate of the Grinberg Point.*

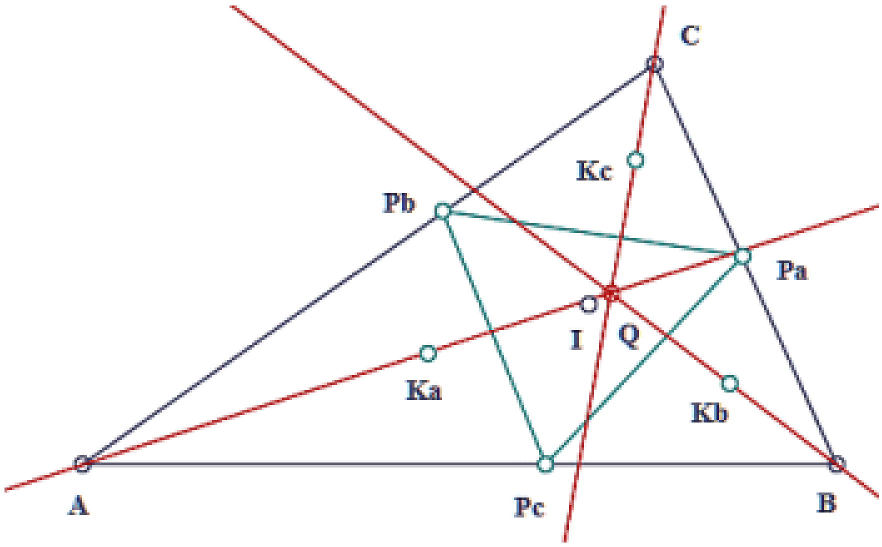

Fig.1.

The computer program "Discoverer" has generalized theorem 1a as follows:

Theorem 2. *The Cevian Corner Product of a finite triangle center P and the Symmedian Point exists, and it is the Isogonal Conjugate of the Complement of the Isotomic Conjugate of P.*

Theorem 2a. *The Cevian Corner Product of a finite triangle center P and the Symmedian Point exists, and it is the Quotient of the Isogonal Conjugate of the Complement of point P divided by the point P.*

In 2003 Darij Grinber has published related result:

Theorem 3 (Grinberg, theorem 1). *The Cevian Corner Product of a finite triangle center P and the Centroid exists, and it is the Complement of the Isotomic Conjugate of P.*

Let n be an integer. We call *n-point* a point with barycentric coordinates (a^n, b^n, c^n). We invite the reader to prove that the following hypothesis is true or false:

Hypothesis. *For any integer n, the Cevian Corner Product of a finite triangle center and the n-point exists.*

In this paper we give a proof of theorem 2 by using barycentric coordinates. Also, we give examples of cevian corner products, discovered by the "Discoverer".

2. Preliminaries

In this section we review some basic facts about barycentric coordinates. We refer the reader to (Grozdev and Nenkov, 2012a,b), (Paskalev & Tchobanov, 1985), (Yiu, 2001, edition of 2013), (Douillet, 2012).

We use barycentric coordinates. The reference triangle *ABC* has vertices $A = (1,0,0)$, $B = (0,1,0)$ and $C = (0,0,1)$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is,

for $\forall k \in \mathbb{R} \setminus \{0\}$: $P = (u, v, w)$ means that $P \simeq (u, v, w) \simeq (ku, kv, kw)$.

A point $P = (u, v, w)$ is finite if $u + v + w \neq 0$. A finite point $P = (u, v, w)$ is *normalized* if $u + v + w = 1$. Given two normalized points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$, then (Paskalev & Tchobanov, 1985, § 15, Proposition 1):

$$|PQ|^2 = -a^2vw - b^2wu - c^2uv, \tag{1}$$

where $u = u_1 - u_2$, $v = v_1 - v_2$ and $w = w_1 - w_2$.

For an arbitrary point $P = (u, v, w)$, the vertices of the cevian triangle of *P* have barycentric coordinates $Pa = (0, v, w)$, $Pb = (u, 0, w)$ and $Pc = (u, v, 0)$. If *P* is a normalized triangle center, the side lengths of $\triangle PaPbPc$ are as follows (Paskalev & Tchobanov, 1985, § 15, Proposition 3):

$$|PbPc|^2 = \frac{a^2vw}{(u+v)(u+w)} + \frac{b^2uw(w-v)}{(u+v)(u+w)^2} + \frac{c^2uv(v-w)}{(u+v)^2(u+w)}, \tag{2}$$

$$|PcPa|^2 = \frac{a^2vw(w-u)}{(u+v)(v+w)^2} + \frac{b^2uw}{(u+v)(v+w)} + \frac{c^2uv(u-w)}{(u+v)^2(v+w)}, \tag{3}$$

$$|PaPb|^2 = \frac{a^2vw(v-u)}{(u+w)(v+w)^2} + \frac{b^2uw(u-v)}{(u+w)^2(v+w)} + \frac{c^2uv}{(u+w)(v+w)} \tag{4}$$

Let *DEF* be a triangle whose vertices have normalized barycentric coordinates wrt $\triangle ABC$ as follows: $D = (p_1, q_1, r_1)$, $E = (p_2, q_2, r_2)$ and $F = (p_3, q_3, r_3)$. Let *P* be a point with normalized barycentric coordinates $P = (p, q, r)$ wrt $\triangle DEF$. Then the barycentric coordinates of $P = (u, v, w)$ wrt $\triangle ABC$ are as follows (Paskalev & Tchobanov, 1985, § 30):

$$\begin{aligned} u &= p_1 p + p_2 q + p_3 r \\ v &= q_1 p + q_2 q + q_3 r \\ w &= r_1 p + r_2 q + r_3 r \end{aligned} \tag{5}$$

The equation of the line joining two points with coordinates (u_1, v_1, w_1) and (u_2, v_2, w_2) is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0 \cdot \tag{6}$$

Three lines $p_i x + q_i y + r_i z = 0$, $i = 1,2,3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \tag{7}$$

The intersection of two lines $L_1 : p_1 x + q_1 y + r_1 z = 0$ and $L_2 :\ p_2 x + q_2 y + r_2 z = 0$ is the point

$$(q_1 r_2 - q_2 r_1, r_1 p_2 - r_2 p_1, p_1 q_2 - p_2 q_1) \,. \tag{8}$$

Given a point $P = (u, v, w)$, the complement of *P* is the point $(v + w, w + u, u + v)$, the isotomic conjugate of *P* is the point (vw, wu, uv) and the isogonal conjugate of *P* is the point $(a^2 vw, b^2 wu, c^2 uv)$.

3. Proof of Theorem 2

Proof. Given $\triangle ABC$. Let $P = (u, v, w)$ be a finite triangle center of $\triangle ABC$ and let $\triangle PaPbPc$ be the cevian triangle of *P*. By using (1), we calculate the side lengths a_1, b_1 and c_1 of $\triangle APcPb$ as follows (see (2) for a_1^2):

$$a_1^2 = |PbPc|^2, \quad b_1 = |APb| = \frac{bw}{u + w}, \quad c_1 = |APc| = \frac{cv}{u + v} \cdot$$

The barycentric coordinates of the symmedian point Ka of $\triangle APcPb$ wrt $\triangle APcPb$ are $Ka = (a_1^2, b_1^2, c_1^2)$. By using (5), we find the barycentric coordinates of $Ka = (uKa, vKa, wKa)$ wrt $\triangle ABC$ as follows:

$$uKa = a^2 vwu^2 + a^2 vw^2 u + a^2 v^2 wu + a^2 v^2 w^2 + 2b^2 w^2 u^2 + 2b^2 w^2 uv$$

$$-b^2wu^2v-b^2wuv^2-c^2u^2vw-c^2uvw^2+2c^2u^2v^2+2c^2uv^2w\,,$$

$$vKa=b^2vw^2(u+v)\,,\quad wKa=c^2v^2w(u+w)\,.$$

The side lengths of $\triangle BPaPc$ are as follows (see (3) for a_2^2):

$$a_2^2=|PcPa|^2\,,\quad b_2=|BPc|=\frac{cu}{u+v}\,,\quad c_2=|BPa|=\frac{aw}{v+w}\,.$$

The barycentric coordinates of the symmedian point *Kb* of $\triangle BPaPc$ wrt $\triangle BPaPc$ are $Kb=(a_2^2,b_2^2,c_2^2)$. By using (5), we find the coordinates of $Kb=(uKb,vKb,wKb)$ wrt $\triangle ABC$ as follows:

$$uKb=a^2uw^2(u+v)\,,$$

$$vKb=-a^2vwu^2-a^2v^2wu+2a^2vw^2u+2a^2v^2w^2+b^2wu^2v+b^2wuv^2$$
$$+b^2w^2u^2+b^2w^2uv+2c^2u^2v^2+2c^2u^2vw-c^2uv^2w-c^2uvw^2\,,$$

$$wKb=c^2u^2w(v+w)$$

The side lengths of $\triangle CPbPa$ are as follows (see (4) for a_3^2):

$$a_3^2=|PaPb|^2\,,\quad b_3=|CPa|=\frac{av}{v+w}\,,\quad c_3=|CPb|=\frac{bu}{u+w}\,.$$

The barycentric coordinates of the symmedian point *Kc* of $\triangle CPbPa$ wrt $\triangle CPbPa$ are $Kc=(a_3^2,b_3^2,c_3^2)$. By using (5), we find the coordinates of $Kc=(uKc,vKc,wKc)$ wrt $\triangle ABC$ as follows:

$$uKc=a^2uv^2(u+w)\,,\quad vKc=b^2u^2v(v+w)\,,$$

$$wKc=2a^2v^2wu+2a^2v^2w^2-a^2vwu^2-a^2vw^2u-b^2wuv^2-b^2w^2uv$$
$$+2b^2wu^2v+2b^2w^2u^2+c^2u^2v^2+c^2uv^2w+c^2u^2vw+c^2uvw^2$$

By using (6), now we find the barycentric equations of the lines *AKa*, *BKb* and *CKc* as follows:

$$AKa:\ c^2v(u+w)y-b^2w(u+v)z=0\,,$$

$$BKb:\ c^2u(v+w)x-a^2w(u+v)z=0\,,$$

$$CKc:\ b^2u(v+w)x-a^2v(u+w)y=0\,.$$

By using (7), we prove that these lines concur in a point. Then, by using (8), we find the point of intersection of the lines *AKa*, *BKb* and *CKc* as the point of intersection $Q=(uQ,vQ,wQ)$ of the lines *AKa* and *BKb*:

$$uQ=a^2vw(u+v)(u+w),\ vQ=b^2wu(v+w)(v+u),\ wQ=c^2uv(w+u)(w+v).$$

Point *Q* is the cevian corner product of point *P* and the symmedian point.

We calculate the isogonal conjugate of the complement of the isotomic conjugate of the point *P*, and we see that this point coincides with point *Q*. This completes the proof.

4. New properties of notable points of the triangle

The computer program "Discoverer" has produced 1218 examples of cevian corner products. Of these 121 are points which are available in (Kimberling) and the rest of 1097 points are not available in (Kimberling). Clearly, the number of examples could be easily extended by the "Discoverer".

We may use the enclosed List K (or equivalently, the enclosed tables Table P-X, or Table X-P) in order to add new theorems to the corresponding articles in the encyclopedias.

Below we give an example. Consider the row 30 of Table X-P. We can rewrite the row to the following theorem:

Theorem 4. *The Euler Reflection Point is the Cevian Corner Product of the Steiner Point and the Symmedian Point.*

Figure 2 illustrates theorem 4. In Fig. 2, *S* is the Steiner point, *PaPbPc* is the cevian triangle of *S*, *Ka, Kb* and *Kc* are the symmedian points of triangles *APcPb*, *BPaPc* and *CPbPa*, respectively. Lines *AKa*, *BKb* and *CKc* concur in the Euler reflection point *E*.

5. New notable points of the triangle

We may use the results in the enclosed List D in order to define new remarkable points of the triangle. We may expect that the cevian corner products available in the List D are new remarkable points, because they are not included in the (Kimberling).

As examples, consider row 32 in List D. The row rewrites to the following theorem:

Theorem 5. *The Cevian Corner Product of the Feuerbach Point and the Symmedian Point exists.*

Figure 3 illustrates theorem 5. In Fig. 3, *F* is the Feuerbach point, *PaPbPc* is the cevian triangle of *F*, *Ka*, *Kb* and *Kc* are the symmedian points of triangles *APcPb*, *BPaPc* and *CPbPa*, respectively. Then lines *AKa*, *BKb* and *CKc* concur in point *Q*, the cevian corner product of the Feuerbach point and the symmedian point.

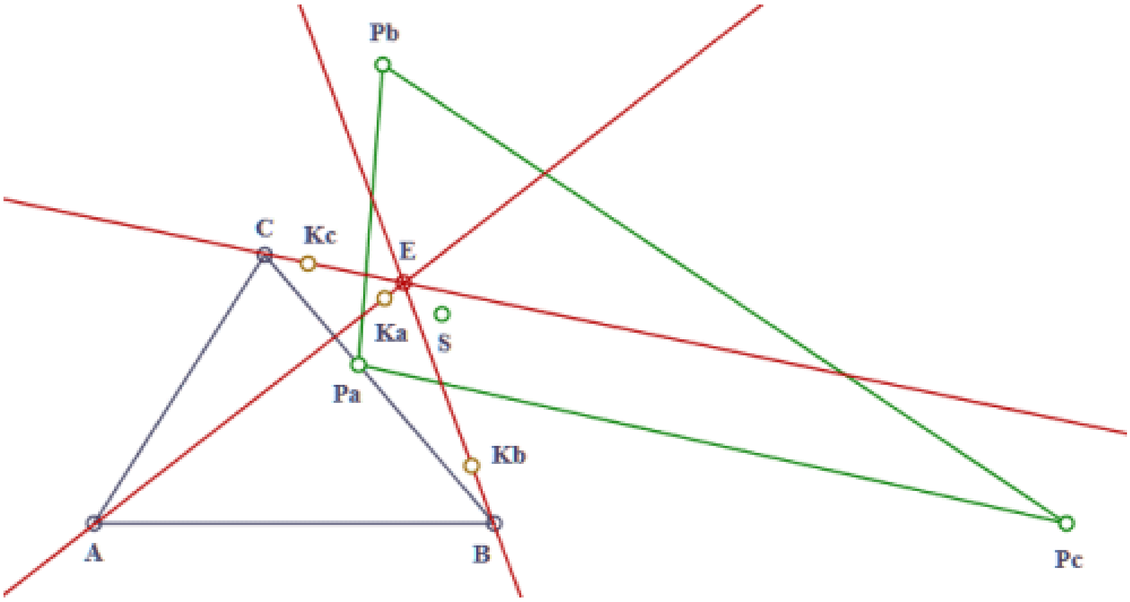

Fig. 2.

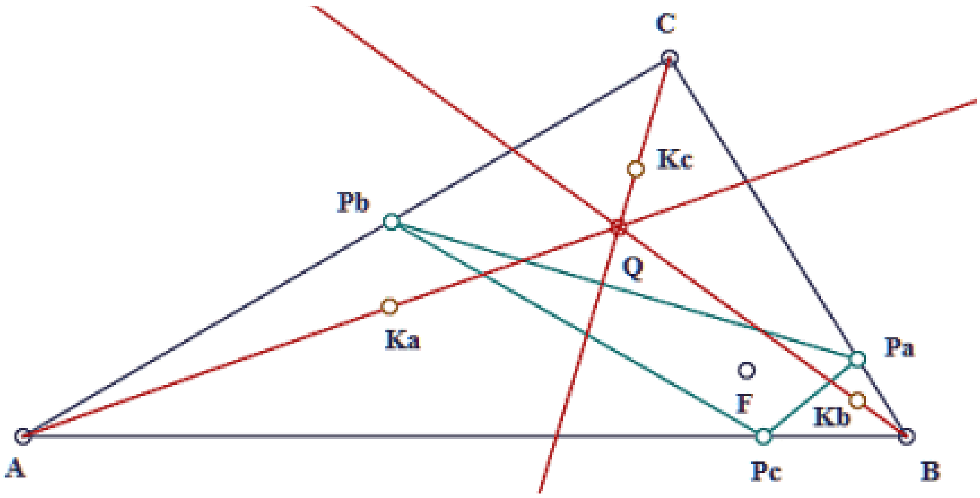

Fig. 3.

We can now define the point "Cevian Corner Product of the Feuerbach Point and the Symmedian Point" as a new remarkable point of the triangle. By using theorem 2, we easily find that the barycentric coordinates of the new point are as follows: $f(a,b,c)$, $f(b,c,a)$ and $f(c,a,b)$, where

$$f(a,b,c) = a^2(c+a-b)(c-a)^2(a+b-c)(a-b)^2$$

$$.((b+c-a)(b-c)^2 + (c+a-b)(c-a)^2).$$

$$.((b+c-a)(b-c)^2 + (a+b-c)(a-b)^2).$$

By using the "Discoverer" we can easily find properties of the new point. Below we give one of the theorems related to the new point:

Theorem 6. *The Cevian Corner Product of the Feuerbach Point and the Symmedian Point is the Ceva Product of the Symmedian Point and the Isogonal Conjugate of the Feuerbach Point.*

The Isogonal Conjugate of the Feuerbach Point is the point X(59) in (Kimberling). Hence, from theorem 6 we conclude that the Ceva Product of points X(6) and X(59) is not included in (Kimberling).

Figure 4 illustrates theorem 6. In Fig. 4, K is the symmedian point, F is the Feuerbach point, gF is the isogonal conjugate of the Feuerbach point, Q is the cevian corner product of the Feuerbach point and the symmedian point, $JaJbJc$ is the anticevian triangle of K. Point A_1 is the intersection of the lines $gFJa$ and BC, B_1 is the intersection of the lines $gFJb$ and CA, C_1 is the intersection of the lines $gFJc$ and AB. Then the lines AA_1, BB_1 and CC_1 concur in point Q. Note that the last three lines are not drown in the figure.

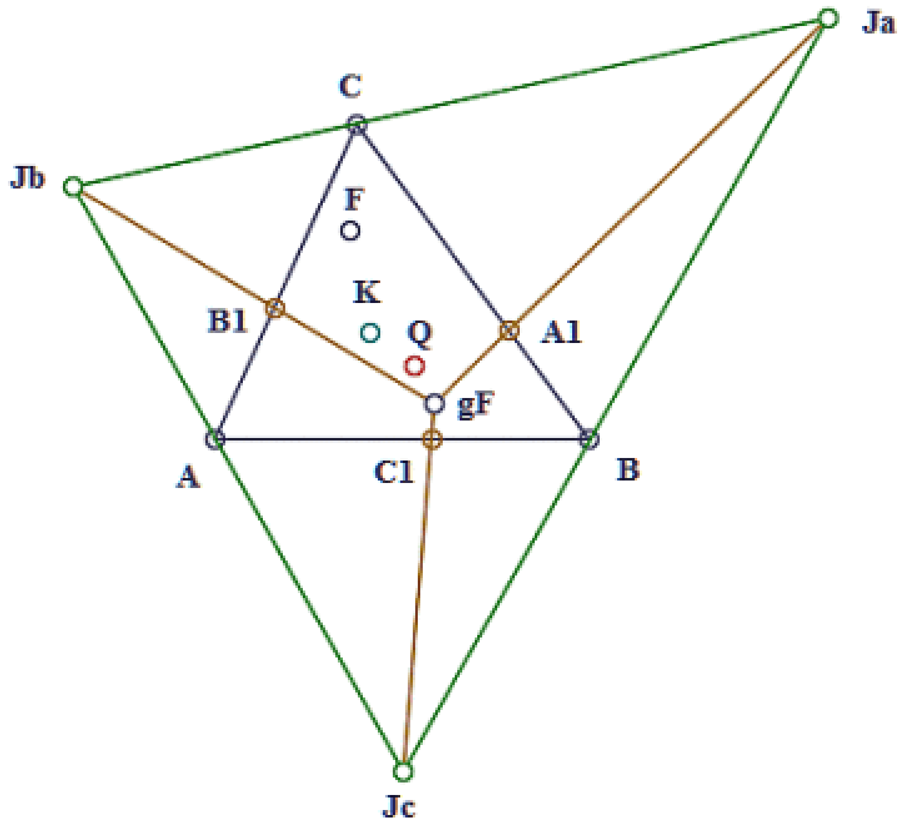

Fig. 4.

6. On a theorem by Pierre Douillet

Pierre Douillet has published the following theorem:

Theorem 7. (Douillet, 17.2.6). *The Triangle of the Incenters of the Cevian Corner Triangles of the Orthocenter is the Triangle of the Orthocenters of the Cevian Corner Triangles of the Gergonne Point.*

By using the "Discoverer" we may find a number of similar theorems. Below we give two theorems:

Theorem 8. *The Triangle of the Incenters of the Cevian Corner Triangles of the Centroid is the Triangle of the Circumcenters of the Cevian Corner Triangles of the Gergonne Point.*

Figure 5 illustrates theorem 8. In Fig. 5, *PaPbPc* is the medial triangle, *QaQbQc* is the intouch triangle, *Oa*, *Ob* and *Oc* are the incenters of triangles *APcPb*, *BPaPc* and *CPbPa*, respectively. Also, *Oa*, *Ob* and *Oc* are the circumcenters of triangles *AQcQb*, *BQaQc* and *CQbQa*, respectively. The circumcircles of triangles *AQcQb*, *BQaQc* and *CQbQa* are not drawn in the figure. Note that triangle *OaObOc* is the Eulet triangle.

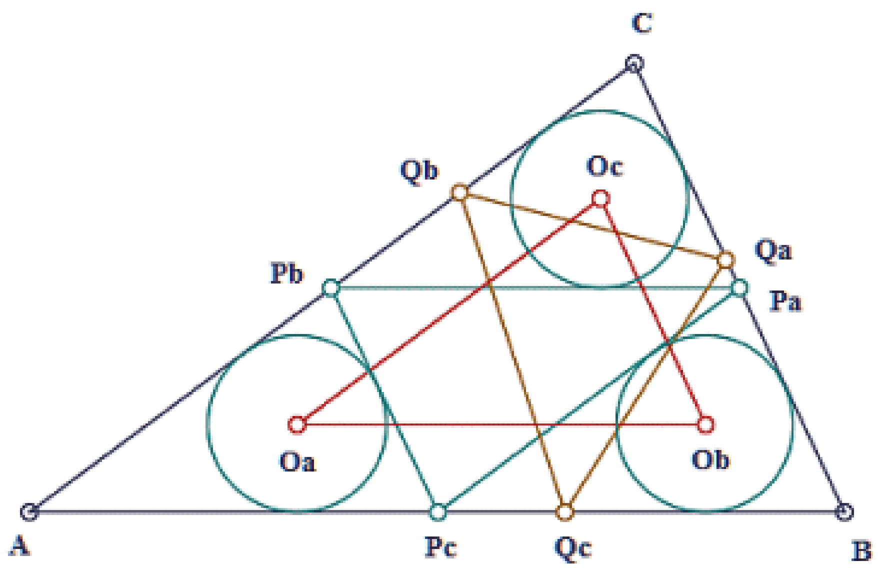

Fig. 5.

Theorem 9. *The Triangle of the Orthocenters of the Cevian Corner Triangles of the Centroid is is the Triangle of the Circumcenters of the Cevian Corner Triangles of the Orthocenter.*

Note that The Triangle of the Orthocenters of the Cevian Corner Triangles of the

Centroid in the above theorem is the Euler triangle of the Incenter. For the definition of the Euler triangle of the Incenter, see (Grozdev & Dekov, 2014b), (Grozdev & Dekov, 2015b, Definitions, Triangles, Euler Triangle).

Supplementary material

The enclosed file “2015_ccp.zip” contains the files quoted in this paper. The reader may download it from the web site of the journal.

REFERENCES

Castellsaguer, Q. The Triangles Web, available at: http://www.xtec.cat/~qcastell/ttw/ttweng/portada.html

Danneels, E. (2003). Some properties of existing points, Hyacinthos message 7892, https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/messages/7892

Grinberg, D. (2003). Isotomcomplement Theory, Hyacinthos message 6423, https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/topics/6423

Douillet, P. (2012). Translation of the Kimberling’s Glossary into barycentrics, available at: http://eg-enc.webege.com/htm/links/glossary.pdf

Grozdev, S. & Dekov, D. (2014a). Computer-generated mathematics: Points on the Kiepert hyperbola, The Mathmatical Gazette, vol. 98, no. 543, 509-511.

Grozdev S. & Dekov D. (2014b). The Computer Program “Discoverer” and the Encyclopedia of Computer-Generated Mathematics” (in Bulgarian), International Journal of Computer-Generated Mathematics, vol 9, no. 2, available at: http://www.ddekov.eu/j/index.htm

Grozdev, S. & Dekov, D. (2015a). A Computer Improves the Steiner’s Construction of the Malfatti Circles, Mathematics and Infomatics, vol. 58, no.1, 40-51.

Grozdev, S. & Dekov, D. (2015b). Computer-Generated Encyclopedia of Euclidean Geometry, in preparation, available at: http://www.ddekov.eu/e2/index.htm

Grozdev, S. & Nenkov, V. (2012a). Three Remarkable Points on the Medians of a Triangle (in Bulgarian), Sofia, Archimedes.

Grozdev, S. & Nenkov, V. (2012b). On the Orthocenter in the Plane and in the Space (in Bulgarian), Sofia, Archimedes.

Kimberling, C., Encyclopedia of Triangle Centers, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

Paskalev, G. & Tchobanov, I. (1985). Remarkable Points in the Triangle (in Bulgarian), Sofa, Narodna Prosveta.

Weisstein, E.W. MathWorld - A Wolfram Web Resource. http://mathworld.wolfram.com/
Yiu, P. (2001, version of 2013). Introduction to the Geometry of the Triangle, available at: http://math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry130411.pdf

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