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RESEARCH ARTICLE

HNN Extensions of Semigroups

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1. Introduction

Let A and B be isomorphic subgroups of a group G and let $\varphi : A \to B$ be an isomorphism. In 1949, G. Higman, B. H. Neumann and H. Neumann [8] defined a group construction now known as the HNN extension of G. They proved that G can be embedded in a group G^* containing an element t such that $t^{-1}at = \varphi(a)$ for all $a \in A$.

In 1963, J. M. Howie [9] defined HNN extensions of semigroups and obtained a result corresponding to the above theorem of Higman, Neumann and Neumann, provided that A and B are unitary subsemigroups of the semigroup S.

Using geometric methods stemming from the work of R. C. Lyndon, J. H. Remmers, and Miller and Schupp, in 1978 D. A. Jackson ([11]; see also [12]) proved again J. M. Howie's result and obtained as well the semigroup analogue of Britton's Lemma.

In 1971, a new approach to HNN extensions of groups is due to J. R. Stallings [16]. J. R. Stallings' approach, based on R. Baer's results on pregroups, yields new proofs of the theorem of Higman, Neumann and Neumann, and Britton's Lemma.

In the present paper we give new proofs of J. M. Howie's and D. A. Jackson's results concerning HNN extensions of semigroups. We use R. Baer's result on presemigroups [2, Section 1, Theorem 1] and J. R. Stallings' method [16, Example 3.A.5.5].

2. Preliminaries

In this section we review some of R. Baer's results on partial semigroups.

A *partial groupoid* is a triple $G = (G, D, \mu)$ where G is a non-empty set, $D \subseteq G \times G$ and $\mu : D \to G$ is a mapping. We denote $\mu(x, y)$ by $x \cdot y$ and use the notations ([14]):

$$(x, y)_D \quad \text{iff} \quad (x, y) \in D \,,$$
$$(x_1, \ldots, x_n)_D \quad \text{iff} \quad (x_i, x_{i+1})_D \quad \text{for } i = 1, \ldots, n-1 \,.$$

Let $G = (G, D, \mu)$ be a partial groupoid and let $F(G)$ be the free semigroup on the set G. If $X = x_1 \ldots x_n \in F(G)$, we say that X is a *word* of *length* n. A word $x_1 \ldots x_n$ is said to be G*-reduced*, if for $i = 1, \ldots, n-1$, $(x_i, x_{i+1}) \notin D$.

Let G be a partial groupoid and let θ_0 be the relation on $F(G)$ defined as follows: $(x \cdot y, xy) \in \theta_0$ iff $(x, y)_D$. Let θ be the congruence on $F(G)$ generated by θ_0. Then $U(G) = F(G)/\theta$ is the *universal semigroup* of the partial groupoid G. The words $X, Y \in F(G)$ are *equivalent*, if $X \sim Y \pmod{\theta}$.

A partial groupoid G is said to be a *partial semigroup* (J.-C. Spehner [15]) if it is embeddable in a semigroup.

Theorem 1. (R. Baer [1]) *A partial groupoid G is a partial semigroup if and only if it embeds in its universal semigroup $U(G)$, i.e., if and only if $x \sim y \pmod{\theta}$ implies $x = y$, for all $x, y \in G$.* ∎

A partial semigroup G is said to be a *presemigroup* if it satisfies the following condition:

(L) Equivalent G-reduced words have the same length.

We introduce the following conditions on a partial groupoid G:

(P1) For all $x, y, z \in G$, if $(x, y, z)_D$, $(x, y \cdot z)_D$ and $(x \cdot y, z)_D$, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

(P2) For all $x, y, z \in G$, if $(x, y, z)_D$, then: $(x, y \cdot z)_D$ if and only if $(x \cdot y, z)_D$.

(P3) For all $w, x, y, z \in G$, if $(w, x, y, z)_D$, then either $(w, x \cdot y)_D$ or $(x \cdot y, z)_D$.

(P4) For all $w, x, y, z \in G$, if $(x, y, z)_D$ and $(w, x \cdot y)_D$, then either $(w, x)_D$ or $(x \cdot y, z)_D$.

(P5) For all $w, x, y, z \in G$, if $(w, x, y)_D$ and $(x \cdot y, z)_D$, then either $(w, x \cdot y)_D$ or $(y, z)_D$.

Theorem 2. (R. Baer [2, Section 1, Theorem 1]) *A partial groupoid G is a presemigroup if and only if it satisfies conditions* (P1) *to* (P5). ∎

Note that recently the author used presemigroups to study semigroup amalgams (see [5], [6]).

3. Unitary HNN extensions of semigroups

We shall use the notation and terminology to be found in Howie [10] or Clifford and Preston [4], to which the reader is referred for basic information on semigroups.

For any semigroup S, S^1 denotes the semigroup obtained from S by adjoining an identity element if it does not already have one.

Let A and B be isomorphic subsemigroups of a semigroup S^1 and let $\varphi : A \to B$ be an isomorphism. The semigroup S^* given by the presentation

$$sgp(S^1, t, t^{-1};\ \ t^{-1}t = tt^{-1} = 1,\ \ t^{-1}at = \varphi(a), \text{ for all } a \in A)$$

is called an *HNN extension of S with stable letter t.*

The product of two elements $s_1, s_2 \in S^1$ will be denoted by $s_1 s_2$ and the inverse of φ will be denoted by φ^{-1}.

Construct three sets in one-to-one correspondence with S^1:

$$S^1, \quad t^{-1}S^1, \quad S^1 t$$

and set $G = S^1 \cup t^{-1}S^1 \cup S^1 t$. Define partial multiplication on G as follows: For all s, s_1, $s_2 \in S^1$ and $b \in B$,

$$
\begin{aligned}
s_1 \cdot s_2 &= s_1 s_2 \ , \\
b \cdot t^{-1}s &= t^{-1}\varphi^{-1}(b)s \ , \\
st \cdot b &= s\varphi^{-1}(b)t \ , \\
s_1 \cdot s_2 t &= s_1 s_2 t \ , \\
t^{-1}s_1 \cdot s_2 &= t^{-1}s_1 s_2 \ , \\
t^{-1}s_1 \cdot s_2 t &= \varphi(s_1 s_2) \ , \text{ iff } s_1 s_2 \in A \ , \\
s_1 t \cdot t^{-1} s_2 &= s_1 s_2 \ .
\end{aligned}
$$

The above defined partial groupoid will be denoted by $G_S = (G, D, \mu)$. The universal semigroup $U(G_S)$ of the partial groupoid G_S is the semigroup given by the presentation

$$sgp(G; xy = z \text{ where } (x,y)_D \text{ and } \mu(x,y) = z)$$

which is another presentation of S^*. Hence S^* is identical with $U(G_S)$.

Recall that a subsemigroup U of a semigroup S is said to be *left unitary* in S (P. Dubreil [7]), if for all $u \in U$ and $s \in S$,

$$us \in U \quad \text{implies} \quad s \in U \ .$$

Right unitariness is defined dually, and U is *unitary* if it is both left and right unitary.

If U is a unitary subsemigroup of S^1, then $1 \in U$ (J. M. Howie [9]).

Lemma 1. *For any semigroup S, any isomorphic subsemigroups A and B of S^1 and any isomorphism $\varphi : A \to B$,*

(i) *G_S satisfies* (P1) *and* (P3).

(ii) *G_S satisfies* (P2) *if and only if B is a unitary subsemigroup of S^1.*

(iii) *G_S satisfies* (P4) *if and only if A is a right unitary subsemigroup of S^1.*

(iv) *G_S satisfies* (P5) *if and only if A is a left unitary subsemigroup of S^1.*

Proof. Let $s, s_1, s_2 \in S^1$, $b \in B$ and $\overline{b} \in S^1 \backslash B$.

(i) A straightforward verification.

(ii) Consider the triplet $(st, b, \overline{b})$. From $(st, b, \overline{b})_D$ and $(st \cdot b, \overline{b}) \notin D$, it follows that the triplet $(st, b, \overline{b})$ satisfies (P2) if and only if $(st, b \cdot \overline{b}) \notin D$, i.e., if and only if B is left unitary in S^1.

Similarly, from $(\overline{b}, b, t^{-1}s)_D$ and $(\overline{b}, b \cdot t^{-1}s) \notin D$, it follows that the triplet $(\overline{b}, b, t^{-1}s)$ satisfies (P2) if and only if $(\overline{b} \cdot b, t^{-1}s) \notin D$, i.e., if and only if B is right unitary in S^1.

A straightforward verification shows that all triplets of elements of G, distinct from $(st, b, \overline{b})$ and $(\overline{b}, b, t^{-1}s)$ satisfy (P2).

Hence G_S satisfies (P2) if and only if B is unitary in S^1.

(iii) Consider the quadruple $(t^{-1}s_1, s_2 t, b, \overline{b})$. Clearly, $(t^{-1}s_1, s_2 t \cdot b)_D$ $\iff s_1 s_2 \varphi^{-1}(b) \in A$. Since $(s_2 t, b, \overline{b})_D$ and $(s_2 t \cdot b, \overline{b}) \notin D$, the quadruple $(t^{-1}s_1, s_2 t, b, \overline{b})$ satisfies (P4) if and only if $(t^{-1}s_1, s_2 t \cdot b)_D$ implies $(t^{-1}s_1, s_2 t)_D$. We have

$$
\begin{aligned}
& \left[(t^{-1}s_1, s_2t \cdot b)_D \Longrightarrow (t^{-1}s_1, s_2t)_D\right] \\
\Longleftrightarrow\quad & \left[s_1s_2\varphi^{-1}(b) \in A \Longrightarrow s_1s_2 \in A\right] \\
\Longleftrightarrow\quad & A \text{ is right unitary in } S^1 .
\end{aligned}
$$

Similarly, the quadruple $(t^{-1}s_1, s_2t, b, st)$ satisfies (P4) if and only if A is right unitary in S^1.

A straightforward verification shows that all quadruples of elements of G distinct from $(t^{-1}s_1, s_2t, b, \overline{b})$ and $(t^{-1}s_1, s_2t, b, st)$ satisfy (P4).

Hence G_S satisfies (P4) if and only if A is right unitary in S^1.

(iv) Consider the quadruple $(\overline{b}, b, t^{-1}s_1, s_2t)$. Clearly, $(b \cdot t^{-1}s_1, s_2t)_D \Leftrightarrow \varphi^{-1}(b)s_1s_2 \in A$. Since $(\overline{b}, b, t^{-1}s_1)_D$ and $(\overline{b}, b \cdot t^{-1}s_1) \notin D$, the quadruple $(\overline{b}, b, t^{-1}s_1, s_2t)$ satisfies (P5) if and only if $(b \cdot t^{-1}s_1, s_2t)_D$ implies $(t^{-1}s_1, s_2t)_D$. We have

$$
\begin{aligned}
& \left[(b \cdot t^{-1}s_1, s_2t)_D \Longrightarrow (t^{-1}s_1, s_2t)_D\right] \\
\Longleftrightarrow\quad & \left[\varphi^{-1}(b)s_1s_2 \in A \Longrightarrow s_1s_2 \in A\right] \\
\Longleftrightarrow\quad & A \text{ is left unitary in } S^1 .
\end{aligned}
$$

Similarly, the quadruple $(t^{-1}s, b, t^{-1}s_1, s_2t)$ satisfies (P5) if and only if A is left unitary in S^1.

A straightforward verification shows that all quadruples of elements of G distinct from $(\overline{b}, b, t^{-1}s_1, s_2t)$ and $(t^{-1}s, b, t^{-1}s_1, s_2t)$ satisfy (P5).

Hence G_S satisfies (P5) if and only if A is left unitary in S^1. Lemma 1 is proved. ∎

Theorem 3 below is equivalent to Theorem 3.5 of [11]. From Theorem 2 and Lemma 1, it follows:

Theorem 3. *Let A and B be isomorphic subsemigroups of S^1 and let $\varphi : A \to B$ be an isomorphism. The partial groupoid G_S is a presemigroup if and only if A and B are unitary subsemigroups of S^1.* ∎

Corollary 1 below is an extension of G. Higman, B. H. Neumann and H. Neumann's theorem ([8, Theorem 1], see also [13, Chap. 4, Section 2]).

Corollary 1. (J. M. Howie [9, Theorem 1]) *Let A and B be isomorphic unitary subsemigroups of S and let $\varphi : A \to B$ be an isomorphism. Then there exists a semigroup containing S^1 and containing also two elements t and t^{-1} such that $tt^{-1} = t^{-1}t = 1$ and, for all $a \in A$, $t^{-1}at = \varphi(a)$.*

Proof. Suppose that S has an identity element. By Theorems 1 and 3, $S = S^1$ embeds in S^*. Now suppose that S does not have an identity element. As in [9], denote $A \cup \{1\}$ by A', and $B \cup \{1\}$ by B'. Extend φ to an isomorphism from A' to B' by defining $\varphi(1) = 1$. Since A and B are unitary subsemigroups of S, by [9, Lemma 1], A' and B' are unitary subsemigroups of S^1. Hence, by Theorems 1 and 3, S^1 embeds in S^*. To complete the proof, identify $t^{-1}1$ with t^{-1} and $1t$ with t. ∎

Corollary 2 below is an extension of Britton's Lemma ([3], see also [13, Chap. 4, Section 2]). From Theorems 1 and 3 it follows:

Corollary 2. *Let A and B be isomorphic unitary subsemigroups of S^1 and let $\varphi : A \to B$ be an isomorphism. If X is a G_S-reduced word distinct from* 1, *then X is not equivalent to* 1. ∎

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