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A numerical method for solving the horizontal resection problem in Surveying

Short Note

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Abstract:

The three-point resection problem, i.e., the problem of obtaining the position of an unknown point from relative angular measurements to three known stations is a basic operation in surveying engineering. In the last centuries, a number of approaches to solve this problem have been developed. In this note, a new numerical approach to solve this problem is presented. The method uses only basic formulae from coordinate geometry. We present also numerical simulations that show the good performance and accuracy of this approach.

Keywords:

numerical algorithms • horizontal resection • Surveys

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1. Introduction

The three-point resection problem, i.e., the problem of obtaining the position of an unknown point from relative angular measurements to three known points (or stations), is a basic operation in surveying engineering (see, e.g. Freudenthal 1968). There are analytic solutions of this problem, e.g. the Kaestner-Burkhardt method (Burtch 2005), also referred to as the Pothonot-Snellius method, the Collins method, the Cassini method, the Tienstra method (for the last three methods see e.g. Burtch 2005; Klinkenberg 1955), the Font-Llagunes and Batlle method (Font-Llagunes and Batlle 2009). All these analytic solutions are suitable for computer implementations and they give fast and precise solutions. But all these methods are relatively complicated, and the descriptions of the methods are relatively long. In this note, we present a relatively simple numerical method, which uses only basic formulae from coordinate geometry. Also, we present simple formulae that allow easy check the solution to the problem.

Additional advantages of the presented numerical method are as follows. The Tienstra method (also known as the barycentric method), has singularities. If the stations are aligned, that is, if the three known points lie on the same line, the method cannot be used, because the reference triangle is degenerate and the barycentric coordinates are not applicable. The Font-Llagunes and Batlle method, as noted by the authors (Font-Llagunes and Batlle 2009), has some problems if the unknown point and two of the known points lie on the same line. The numerical method, presented is this paper, does not have any singularities, except for the intrinsic singularities of the three-point resection problem. In the cases when the Tienstra and Font-Llagunes-Batlle methods have problems, the presented numerical method works successfully.

Although the analytic methods are faster than the presented numerical method, the presented numerical method is fast enough for computer implementation. The numerical method allows the solution to be found for less than one second, provided we use a desktop personal computer. We present numerical simulations that show the good performance and accuracy of this approach. The method is simple, so that it allows a simple implementation. We have created a simple computer program by using the programming language PHP. The computer program is used in the numerical simulations given below, as well as in many other additional simulations which are not included in this note.

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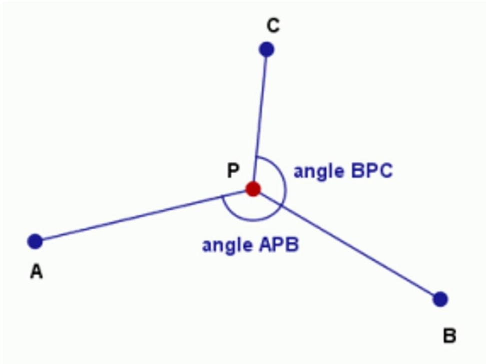

Figure 1. The three point resection problem.

Recall that the three-point resection problem has intrinsic singularities. The problem has infinitely many solutions, if the four points, that is, the three known points and the unknown point, lie on the same circle or on the same line. Any computer program has to detect these cases and to inform the user by a suitable message. Also, note that the three-point resection problem requires that no two of the four points coincide.

2. The method

The three-point resection problem is as follows. Given three points $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$ and two angles, find the coordinates of a point P such that $\angle APB$ and $\angle BPC$ are respectively equal to the given angles (See Figure 1).

Given any vectors a and b, we denote by $a.b$ the dot product of the vectors and by $|a|$ the length of the vector a. Let $\vec{PA}$, $\vec{PB}$ and $\vec{PC}$ are vectors from point P to points A, B and C, respectively. We use the well known formulae

$$cos(\angle APB) = \frac{\vec{PA}.\vec{PB}}{|\vec{PA}|.|\vec{PB}|}, \tag{1}$$

$$cos(\angle BPC) = \frac{\vec{PB}.\vec{PC}}{|\vec{PB}|.|\vec{PC}|}. \tag{2}$$

From the above formulae we obtain the system

$$\left| \begin{array}{l} cos(\angle APB).|\vec{PA}|.|\vec{PB}| - \vec{PA}.\vec{PB} = 0 \\ cos(\angle BPC).|\vec{PB}|.|\vec{PC}| - \vec{PB}.\vec{PC} = 0 \end{array} \right. \tag{3}$$

Let point P has coordinates x_1 and x_2. We obtain the following system for x_1 and x_2:

$$\left| \begin{array}{ll} cos(\angle APB) & \sqrt{(x_1-a_1)^2+(x_2-a_2)^2} \\ & \sqrt{(x_1-b_1)^2+(x_2-b_2)^2} \\ & - \ [(x_1-a_1)(x_1-b_1)+(x_2-a_2)(x_2-b_2)] = 0 \\ cos(\angle BPC) & \sqrt{(x_1-b_1)^2+(x_2-b_2)^2} \\ & \sqrt{(x_1-c_1)^2+(x_2-c_2)^2} \\ & - \ [(x_1-b_1)(x_1-c_1)+(x_2-b_2)(x_2-c_2)] = 0 \end{array} \right. \tag{4}$$

Denote by $f_1(x_1, x_2)$ and by $f_2(x_1, x_2)$ the left side of the first and second of the above equations, respectively. Define $f(x_1, x_2) = |f_1(x_1, x_2)|^2 + |f_2(x_1, x_2)|^2$. Suppose that a root of the function $f(x_1, x_2)$ is within the rectangle $[a, b] \times [c, d]$. We divide the segments $[a, b]$ and $[c, d]$ by N equal parts by using the points $p_0 = a, p_1, p_2, \cdots, p_N = b$ and $q_0 = c, q_1, q_2, \cdots, q_N = d$. Then we evaluate $f(x_1, x_2)$ for each x_1 in $\{p_0, p_1, p_2, \cdots, p_N\}$ and x_2 in $\{q_0, q_1, q_2, \cdots, q_N\}$, and we select the minimal of these values. We use the minimal value as the center of a new rectangle whose sides have 10 times smaller length than the sides of the previous rectangle. We use 10 times smaller length of the sides of the new rectangle, since we use the decimal system for numbers, and in this case each new step adds one true digit to the answer. The process is repeated until the root is found. (See Figure 2). From formula 1 we obtain a formula which we use to calculate the value of $\angle APB$:

$$\angle APB = acos\left(\frac{\vec{PA}.\vec{PB}}{|\vec{PA}|.|\vec{PB}|}\right). \tag{5}$$

We use similar formulae to calculate the other two angles. Note that these formulae are suitable for check of the answer, if we use any numerical or analytic method to solve the three-point resection problem. At each step n we calculate the following sum:

$$|\angle APB - \angle APBn| + |\angle BPC - \angle BPCn| + |\angle CPA - \angle CPAn|, \tag{6}$$

where $\angle APB$, $\angle BPC$ and $\angle PCA$ are the input angles and $\angle APBn$, $\angle BPCn$ and $\angle CPAn$ are the corresponding angles calculated at this step. If the above sum becomes equal to 0, the computer program stops the calculations, since the root is found. The method works well, if $N \geq 50$. In the computer program, used to solve the numerical simulations given below, we set $N = 50$. Note that at each step the number of calculations is the same, but we search point P within a rectangle whose area is 100 times smaller than the area of the preceding rectangle.

3. Numerical simulations

In simulation 1 below we use the same input data, as used by Burtch (Burtch 2005) to illustrate the use of the Kaestner-Burkhardt, Cassini, Collins and Tienstra methods.

Simulation 1. Given points A(1000,5300), B(3100,5000) and C(2200,6300). Find a point P such that $\angle BPC = 115° \ 05' \ 20''$ and $\angle CPA = 109° \ 30' \ 45''$.

Solution. Since $\angle BPC = 115° \ 05' \ 20'' = 115.0889°$ and $\angle CPA = 109° \ 30' \ 45'' = 109.5125°$, we obtain $\angle APB = 135.3986°$. Since the unknown point P is inside into the $\triangle ABC$, the computer program selects as initial square for finding P the square $KLMN = [900, 3200] \times [4500, 6800]$ which contains the triangle. Computer program needs 6 steps to receive the following coordinates of point $P(x_1, x_2)$: $x_1 = 2128.39044$ and $x_2 = 5578.14432$. (See Figure 2).

We could record the calculations, made by the computer in the above simulation. The file containing a record of calculations is

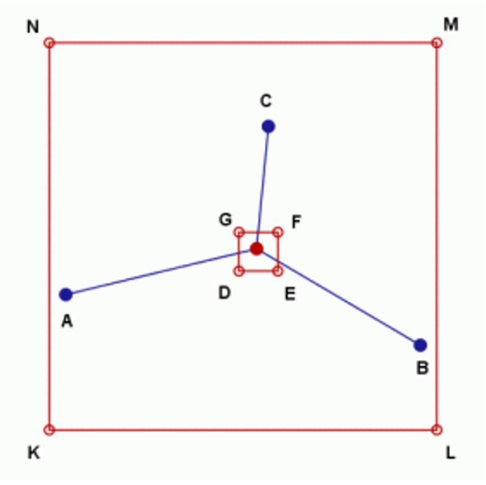

Figure 2. Solution of the three-point resection problem: As a first step, we search point P (the point within the rectangle $DEFG$) within a rectangle where point P is located - rectangle $KLMN$ in the figure. As a second step, we search point P in a rectangle whose area is 100 times smaller than the area of the preceding rectangle - rectangle $DEFG$ in the figure.

Table 1.

Step	1	2	3	4	5	6
Error	4.6248	0.3244	0.0451	0.0015	0.0004	0

available for download as supplementary material. In Table 1 is shown the error, that is the sum 6, calculated at each step.

In Simulation 2 below we use the same input data (slightly rounded) as in the simulation 2 given in the paper by Font-Llagunes and Batlle (2009).

Simulation 2. Given points $A(0,10)$, $B(-8.6603,-5)$ and $C(8.6603,-5)$. Find a point P such that $\angle APB = 82.4028°$ and $\angle BPC = 180°$.

Solution. Since $\angle BPC$ is 180°, the computer program concludes that point P is on the segment BC. Hence, the computer program limits the search of point P only within the line segment which is the intersection of line BC and square $KLMN = [-10,10] \times [-15,5]$. Square $KLMN = [-10,10] \times [-15,5]$ is selected by the computer program as initial square for search of point P, since it contains the segment BC. In this case, the computer program just sets $x_2 = -5$ for the second coordinate of point P and performs search only for the first coordinate. The computer program needs 5 steps in order the error, that is the sum 6, to become 0. At step 5 we receive the following coordinates of point $P(x_1, x_2)$: $x_1 = 2.00068$ and $x_2 = -5.00000$. More details are given in the supplementary file.

By using the computer program, we have performed a number of other simulations. All simulations prove that the method works efficiently and correctly.

4. Conclusions

In this paper, a new numerical solution to the three-point resection problem has been presented. This method represents an alternative to the other existing solutions of the problem. This method uses only basic formulae from coordinate geometry. By means of numerical simulations using a simple computer program, good performance of the method has been proven. The computer program works fast and correctly. The present method is simpler than the analytic methods and it does not have any singularities, except for the intrinsic singularities of the three-point resection problem. Other approaches suffer from more singularities that make the unknown point unreachable, to name one, the Tienstra method is undetermined when the three stations are aligned. Also, we offer formulae for the check of the answer, which are possibly the simplest ones for the check of the answer of the three-point resection problem, if we use any numerical or analytic method.

5. Acknowledgement

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