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Vol. 141, No. 1, August 1, 1991
Printed in Belgium

The Embedding of Semigroup Amalgams

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Communicated by T. E. Hall

Received June 23, 1989

In 1927 O. Schreier [19] showed that a group amalgam is always embeddable in a group. In 1957 N. Kimura ([15]; see also [4]) showed that a semigroup amalgam cannot always be embedded in a semigroup. The semigroup amalgams were first studied extensively by J. M. Howie ([8–13]; see also [4, 14]). In 1975 T. E. Hall [6] showed that Schreier's theorem extends to the class of inverse semigroups. G. B. Preston [16] has shown that T. E. Hall's method yields a new proof of Howie's result concerning amalgamation over unitary subsemigroups. Short proofs of results in [16] are due to T. E. Hall [7]. In the present paper we give a new proof of J. M. Howie's result concerning amalgamation over unitary subsemigroups. This proof is based on R. Baer's results [1–3].

1. PRELIMINARIES

In this section we review some basic facts about partial semigroups.

A *partial groupoid* is a triple $G=(G, D, \mu)$, where $D \subseteq G \times G$ and $\mu: D \to G$ is a mapping. We use the notations [21]

$$xy = \mu(x, y)$$

$$(x, y)_D \qquad \text{iff} \quad (x, y) \in D.$$

Let $G=(G, D, \mu)$ be a partial groupoid and $F(G)$ be the free semigroup on the set G. The element $X=(x_1, ..., x_n) \in F(G)$ is called a *word* of *length* n. The word $(x_1, ..., x_n)$ is said to be *reduced* if for $i=1, ..., n-1$, $(x_i, x_{i+1}) \notin D$. We use the notations [17, 18]

$$(x_1, ..., x_n)_D \qquad \text{iff} \quad (x_i, x_{i+1})_D \text{ for } i=1, ..., n-1.$$

Let θ_0 be the relation on $F(G)$ defined by $(xy, (x, y)) \in \theta_0$ iff $(x, y)_D$. Let θ be the congruence on $F(G)$ generated by θ_0. Then $U(G)=F(G)/\theta$ is the

0021-8693/91 $3.00

universal semigroup of the partial groupoid G. The words X and Y are *equivalent* if $X \sim Y \pmod{\theta}$.

DEFINITION 1 [20, Definition 1]. A partial groupoid $G=(G, D, \mu)$ is said to be *partial semigroup* if it is embeddable in a semigroup.

DEFINITION 2. A partial semigroup $G=(G, D, \mu)$ is said to be *presemigroup* if it satisfies the following property:

(L) [3, Property (L)]: Equivalent reduced words have the same length.

We introduce the following notations:

(P1) For all $x, y, z \in G$, if $(x, y, z)_D$, $(x, yz)_D$, and $(xy, z)_D$, then $x(yz)=(xy)z$.

(P2) For all $x, y, z \in G$, if $(x, y, z)_D$, then $(x, yz)_D$ if and only if $(xy, z)_D$.

(P3) For all $w, x, y, z \in G$, if $(w, x, y, z)_D$, then either $(w, xy)_D$ or $(xy, z)_D$.

(P4) For all $w, x, y, z \in G$, if $(x, y, z)_D$ and $(w, xy)_D$, then either $(w, x)_D$ or $(xy, z)_D$.

(P5) For all $w, x, y, z \in G$, if $(w, x, y)_D$ and $(xy, z)_D$, then either $(w, xy)_D$ or $(y, z)_D$.

THEOREM 1 (R. Baer [3, Section 1, Theorem 1]). *A partial groupoid $G=(G, D, \mu)$ is a presemigroup if and only if it satisfies axioms* (P1) *to* (P5).

2. EMBEDDING UNITARY AMALGAMS

A *semigroup amalgam* consists of a semigroup U together with a family $\{S_i : i \in I\}$ of semigroups and a family $\{\varphi_i : i \in I\}$ of monomorphisms $\varphi_i : U \to S_i$. We denote the semigroup amalgam by

$$\mathscr{A} = [\{S_i : i \in I\};\ U;\ \{\varphi_i : i \in I\}].$$

The semigroup amalgam $\mathscr{A}$ determines a partial groupoid $\mathscr{G}_{\mathscr{A}}$ and the amalgam is said to be *embeddable* in a semigroup if $\mathscr{G}_{\mathscr{A}}$ is a partial semigroup [4, Section 9.4]. As in [4], write $U_i = U\varphi_i$, $S_i' = (S_i \setminus U_i) \cup U$ and $G = \bigcup \{S_i' : i \in I\}$. We denote $\mathscr{G}_{\mathscr{A}} = (G, D)$.

LEMMA 1. *For any semigroup amalgam* $\mathscr{A} = [\{S_i : i \in I\}; U; \{\varphi_i : i \in I\}]$, *the partial groupoid* $\mathscr{G}_{\mathscr{A}}$ *satisfies axioms* (P3), (P4), *and* (P5).

Proof. To prove that (P3) holds, suppose $w, x, y, z \in G$, $(w, x, y, z)_D$ and $x, y \in S_i'$ for some $i \in I$. If $xy \in U$, then $(w, xy)_D$ and $(xy, z)_D$. Thus (P3) holds. Now suppose $xy \in S_i' \setminus U$. Then either $x \in S_i' \setminus U$ or $y \in S_i' \setminus U$. If $x \in S_i' \setminus U$, since $(w, x)_D$, we have $w \in S_i'$. Therefore $(w, xy)_D$. Similarly, if $y \in S_i' \setminus U$, then $(xy, z)_D$. Thus (P3) holds.

To prove that (P4) holds, suppose $w, x, y, z \in G$, $(x, y, z)_D$, $(w, xy)_D$ and $x, y \in S_i'$ for some $i \in I$. If $xy \in U$, then $(xy, z)_D$. Thus (P4) holds. Now suppose $xy \in S_i' \setminus U$. Since $(w, xy)_D$, we have $w \in S_i'$. Therefore $(w, x)_D$. Thus (P4) holds.

Similarly, (P5) holds. Lemma 1 is proved.

DEFINITION 3 (P. Dubreil [5]). A subsemigroup U of a semigroup S is said to be *unitary* in S if, for all $u \in U$, $s \in S$,

(i) $us \in U$ implies $s \in U$.

(ii) $su \in U$ implies $s \in U$.

THEOREM 2. *Let* $\mathscr{A} = [\{S_i : i \in I\}; U; \{\varphi_i : i \in I\}]$ *be a semigroup amalgam. If U is a unitary subsemigroup of* S_i' *for all* $i \in I$, *then* $\mathscr{G}_{\mathscr{A}}$ *is a presemigroup.*

Proof. By Theorem 1, the partial groupoid $\mathscr{G}_{\mathscr{A}}$ is a presemigroup if and only if it satisfies axioms (P1) to (P5). By Lemma 1, $\mathscr{G}_{\mathscr{A}}$ satisfies (P3), (P4), and (P5). We prove that $\mathscr{G}_{\mathscr{A}}$ satisfies (P1) and (P2).

Suppose that $x, y, z \in G$, $(x, y, z)_D$ and $x, y \in S_i'$ for some $i \in I$. If $z \in S_i'$, then x, y and z belong to the same semigroup. Thus (P1) and (P2) hold. Now suppose $z \notin S_i'$, i.e., $z \in S_j' \setminus U$ and $i \neq j$. Since $(y, z)_D$, we have $y \in U$. If $x \in U$, then x, y, and z belong to the same semigroup. Thus (P1) and (P2) hold. If $x \in S_i' \setminus U$, since U is unitary in S_i', we have $xy \in S_i' \setminus U$, and since U is unitary in S_j', we have $yz \in S_j' \setminus U$. Consequently $(xy, z) \notin D$ and $(x, yz) \notin D$. Thus (P1) and (P2) hold. This completes the proof of Theorem 2.

As a corollary to Theorem 2 we have

COROLLARY [8]. *Let* $\mathscr{A} = [\{S_i : i \in I\}; U; \{\varphi_i : i \in I\}]$ *be a semigroup amalgam. If U is a unitary subsemigroup of* S_i' *for all* $i \in I$, *then* $\mathscr{G}_{\mathscr{A}}$ *is a partial semigroup.*

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium