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THE CLASS OF ALL S-PREGROUPS IS NOT FINITELY AXIOMATIZABLE

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ABSTRACT. In order to investigate the amalgamated free products of groups, in 1950 R. Baer (*Free sums of groups and their generalizations*. II, Amer. J. Math. **72** (1950), 625–646) introduced the concept of an S-pregroup and gave an infinite set of elementary (i.e., of a first-order language) axioms for S-pregroups. The term "S-pregroup" was introduced by J. R. Stallings (*Adian groups and pregroups*, Essays in Group Theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer-Verlag, New York, 1987, pp. 321–342), who suggested the problem of finding a finite set of elementary axioms for S-pregroups (ibid, Question 5, The first part, p. 340). In the present paper we show that the class of all S-pregroups is not finitely axiomatizable, i.e., it cannot be characterized by any finite set of elementary axioms.

1. PRELIMINARIES

In this section we review some of R. Baer's results on S-pregroups.

A *partial groupoid* is a triple $G = (G, D, \mu)$ where G is a nonempty set, $D \subseteq G \times G$ and $\mu: D \to G$ is a mapping. We use the following notations [5, 4]:

$$x \cdot y = \mu(x, y),$$
$$(x, y)_D \quad \text{iff} \quad (x, y) \in D.$$

Let $G = (G, D, \mu)$ be a partial groupoid and $F(G)$ be the free group on the set G. Let θ_0 be the relation on $F(G)$ defined as follows: $(x \cdot y, xy) \in \theta_0$ iff $(x, y)_D$. Let θ be the congruence on $F(G)$ generated by θ_0. Then $U(G) = F(G)/\theta$ is the *universal group* of the partial groupoid G.

Definition. A partial groupoid $G = (G, D, \mu)$ is said to be an *S-pregroup* if it satisfies the following axioms:

(A1) There exists an identity element $1 \in G$ such that for all $x \in G$, $(x, 1)_D$, $(1, x)_D$, and $x \cdot 1 = 1 \cdot x = x$.

(A2) For each $x \in G$, there exists $x^{-1} \in G$ such that $(x, x^{-1})_D$, $(x^{-1}, x)_D$, and $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

(A3) For all $x, y, z \in G$,

(i) if $(x, y)_D$ and $(y, z)_D$, then: $(x \cdot y, z)_D$ iff $(x, y \cdot z)_D$.

(ii) if $(x, y)_D$, $(y, z)_D$, and $(x \cdot y, z)_D$, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

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(A4) For all $x, y \in G$, $x \sim y \pmod{\theta}$ implies $x = y$.

(A5) For each reduced word X, $1 \sim X \pmod{\theta}$ implies $1 = X$.

Axiom (A4) means that G embeds in its universal group. Axiom (A5) means that in the universal group of G the identity element cannot be represented as a nontrivial reduced word.

Theorem 1 (R. Baer [1, §4, Theorem]). *A partial groupoid $G = (G, D, \mu)$ is an S-pregroup if and only if it satisfies axioms* (A1), (A2), (A3), *and for each $n \geq 4$, the following axiom*:

(S_n) *For all $x_1, \ldots, x_n \in G$, if for $i = 1, \ldots, n-1$, $(x_i, x_{i+1}^{-1})_D$ and $(x_n, x_1^{-1})_D$, then there exists $j \leq n-2$ such that $(x_j, x_{j+2}^{-1})_D$.*

It is routine to verify that the infinite set of axioms for S-pregroups due to R. Baer can be written as an infinite set of elementary axioms.

2. The class of all S-pregroups is not finitely axiomatizable

We will prove that the infinite set of elementary axioms for S-pregroups due to R. Baer is not equivalent to any of its finite subsets. From this follows the fact that the class of all S-pregroups is not finitely axiomatizable. The reader is referred to [2] for the fundamentals of first-order theories.

Theorem 2. *The class of all S-pregroups is not finitely axiomatizable.*

Proof. By Theorem 1, a partial groupoid G is an S-pregroup if and only if it satisfies axioms (A1), (A2), (A3), and for each $n \geq 4$, axiom (S_n). We will prove that (A1), (A2), (A3), and (S_k) for $k = 4, 5, \ldots, n-1$ do not imply (S_n). To prove this we construct a partial groupoid G_n that satisfies (A1), (A2), (A3), and (S_k) for $k = 4, 5, \ldots, n-1$ but not (S_n).

Let $G_n = (G_n, D, \mu)$. We define

$$G_n = \{1, a_0, \ldots, a_{n-1}, a_0^{-1}, \ldots, a_{n-1}^{-1}, b_0, \ldots, b_{n-1}, b_0^{-1}, \ldots, b_{n-1}^{-1}\}.$$

We will denote by $\oplus$ addition modulo n. Let $i = 0, 1, \ldots, n-1$. We define partial multiplication in G_n as follows: 1 is the identity of G_n and $a_i^{-1}[b_i^{-1}]$ is the inverse of a_i $[b_i]$. All other products are

$$b_i \cdot a_{i\oplus 1} = a_i, \qquad b_i^{-1} \cdot a_i = a_{i\oplus 1}, \qquad a_i \cdot a_{i\oplus 1}^{-1} = b_i,$$
$$a_{i\oplus 1}^{-1} \cdot b_i^{-1} = a_i^{-1}, \qquad a_i^{-1} \cdot b_i = a_{i\oplus 1}^{-1}, \qquad a_{i\oplus 1} \cdot a_i^{-1} = b_i^{-1}.$$

Clearly, G_n satisfies (A1) and (A2). It is routine to verify that G_n satisfies (A3). We will prove that G_n satisfies (S_k) for $k = 4, 5, \ldots, n-1$.

Let $X = x_1 \cdots x_m$ be a word. A *subword* of X is a word $x_r x_{r+1} \cdots x_s$, where $1 \leq r \leq s \leq m$. The subword $x_1 \cdots x_s$ (i.e., $r = 1$) of X is called a *left factor* of X [3], and the element x_1 is called the *first letter* of X. We will say that the word $x_1 \cdots x_m$ is *regular* if for $i = 1, \ldots, m-1$, $(x_i, x_{i+1}^{-1})_D$ and $(x_m, x_1^{-1})_D$. Thus (S_m), $m \geq 4$, holds iff for every regular word $x_1 \cdots x_m$ there exists $j \leq m-2$ such that $(x_j, x_{j+2}^{-1})_D$.

We will consider regular words of length k, where $4 \leq k \leq n-1$.

If in a regular word $X = x_1 \cdots x_k$, we have $x_i = 1$ for some $i \leq k$ or $x_j = x_{j+1}$ for some $j \leq k-1$, then, obviously, X satisfies (S_k). In what follows we will omit these cases.

For $x \in G_n$ we define

$$C(x) = \{y \colon (x, y^{-1})_D,\ y \neq 1,\ \text{and } y \neq x\}.$$

For $i = 0, 1, \dots, n-1$, we have

$$C(a_i) = \{a_{i\ominus 1}, a_{i\oplus 1}\}, \tag{1}$$

$$C(a_i^{-1}) = \{b_{i\ominus 1}, b_i^{-1}\}, \tag{2}$$

$$C(b_i) = \{a_{i\oplus 1}^{-1}\}, \tag{3}$$

$$C(b_i^{-1}) = \{a_i^{-1}\}. \tag{4}$$

We distinguish four cases.

Case 1. The first letter of a regular word X is a_i. Using (1) we see that X has as a subword at least one of the following words:

$$a_j a_{j\ominus 1} a_j \quad \text{or} \quad a_j a_{j\oplus 1} a_j, \quad \text{where } j \in \{0, 1, \dots, n-1\}.$$

Thus (S_k) holds.

Case 2. The first letter of a regular word X is a_i^{-1}. Using (2), (3), and (4) we see that X has as a left factor, either

$$a_i^{-1} b_{i\ominus 1} a_i^{-1} \quad \text{or} \quad a_i^{-1} b_i^{-1} a_i^{-1}.$$

Thus (S_k) holds.

Case 3. The first letter of a regular word X is b_i. Using (3), (2), and (4) we see that X has as a left factor, either

$$b_i a_{i\oplus 1}^{-1} b_i \quad \text{or} \quad b_i a_{i\oplus 1}^{-1} b_{i\oplus 1}^{-1} a_{i\oplus 1}^{-1}.$$

Thus (S_k) holds.

Case 4. The first letter of a regular word X is b_i^{-1}. Using (4), (2), and (3) we see that X has as a left factor, either

$$b_i^{-1} a_i^{-1} b_{i\ominus 1} a_i^{-1} \quad \text{or} \quad b_i^{-1} a_i^{-1} b_i^{-1}.$$

Thus (S_k) holds.

We have proved that G_n satisfies (S_k) for $k = 4, 5, \dots, n-1$. To show that G_n does not satisfy (S_n), note that the word $a_0 a_1 \cdots a_{n-1}$ is regular, but for $j = 0, 1, \dots, n-3$, $(a_j, a_{j+2}^{-1}) \notin D$. This completes the proof of Theorem 2.

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