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# The class of all embeddable semigroup amalgams is not finitely axiomatizable

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**Abstract**

We prove that the class of all embeddable semigroup amalgams is not first-order finitely axiomatizable. © 1998 Elsevier Science B.V.

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## 1. Introduction

In 1927, Schreier [23] showed that a group amalgam is always embeddable in a group. In 1956, Jónsson [18, p. 206] gave an example to show that a semigroup amalgam need not be embeddable in a semigroup. In 1957, Kimura [19] (see also [3, Section 9.4]) showed that a commutative semigroup amalgam need not be embeddable in a semigroup.

Semigroup amalgams were first extensively studied by Howie [12–17] (see also [3, 11]). In 1975, Hall [9] showed that Schreier's theorem extends to the case of inverse semigroups.

In 1975, Lallement [20] gave a necessary and sufficient condition for a semigroup amalgam $\mathscr{A} = [\{S_i: i \in I\}; U]$ to be embeddable in a semigroup. Lallement's condition is in the form of a countable set of equational implications with existential quantifiers and with variables taken from card $I$ distinct sets. Lallement has shown that no finite set of such implications can serve as a necessary and sufficient condition for a semigroup amalgam to be embeddable.

In the present paper we prove (Theorem 1) that the class of all embeddable semigroup amalgams is not first-order finitely axiomatizable. In the proof of Theorem 1 we construct partial groupoids $G_S$ and $G_T$. The union of the sets of multiplications of $G_S$ and $G_T$ is exactly the "length $n$ zigzag of type I over $U$ from $s \in S$ to $t \in T$ with spine $u_1, v_1, \ldots, u_n, v_n, u_{n+1}$" as defined by Hall [10, Section 8, Definition 5].


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## 2. Preliminaries

A *partial groupoid* is a triple $G = (G, D, \mu)$, where $G$ is a nonempty set, $D \subseteq G \times G$, and $\mu : D \to G$ is a mapping. Let $G = (G, D, \mu)$ and $H = (H, E, \nu)$ be partial groupoids. Denote $\mu(x, y)$ by $x \cdot y$ and $\nu(x, y)$ by $x * y$. A *homomorphism* of $G$ into $H$ is a mapping $\varphi : G \to H$ such that $(x, y) \in D$ implies $(\varphi(x), \varphi(y)) \in E$ and $\varphi(x \cdot y) = \varphi(x) * \varphi(y)$, for all $x, y \in G$.

Let $G = (G, D, \mu)$ be a partial groupoid and let $G^+$ be the free semigroup on the set $G$. Let $\theta_0(G)$ be the relation on $G^+$ consisting of the pairs $(xy, \mu(x, y))$ for all $(x, y) \in D$. Let $\theta(G)$ be the congruence on $G^+$ generated by $\theta_0(G)$. Then $U(G) = G^+/\theta(G)$ is the *universal semigroup* of the partial groupoid $G$. For each word $w$ of $G^+$ we denote by $w\theta$ the $\theta(G)$-class of $w$.

Let $G$ and $H$ be partial groupoids. $G$ is said to be *embeddable* into $H$ if there is a one-to-one homomorphism of $G$ into $H$. A partial groupoid $G$ is said to be a *partial semigroup* [24] if it is embeddable in a semigroup.

**Theorem A** (Baer [1]). *A partial groupoid $G$ is a partial semigroup if and only if it embeds in its universal semigroup $U(G)$, i.e., if and only if $x \sim y$ $(\mathrm{mod}\, \theta(G))$ implies $x = y$, for all $x, y \in G$.*

Note that recently partial semigroups were studied by the author [5–8].

We shall examine the congruence $\theta(G)$ by considering elementary $\theta_0(G)$-transitions as defined in [11, Ch. I, Section 5]. If $x, y \in G$, then $x \sim y$ $(\mathrm{mod}\, \theta(G))$ if and only if either $x = y$ or for some $n \geq 1$ there is a sequence

$$x = w_0 \to w_1 \to \cdots \to w_n = y$$

of elementary $\theta_0(G)$-transitions connecting $x$ to $y$ ([11, Proposition I.5.10]). For an arbitrary sequence of elementary $\theta_0(G)$-transitions we say that the number of elementary $\theta_0(G)$-transitions in the sequence is the *length* of the sequence.

Let $G = (G, D, \mu)$ be a partial groupoid and let $0 \notin G$. We extend the multiplication of $G$ to one of $G^0 = G \cup \{0\}$ by defining $x \circ y = \mu(x, y)$ for all $(x, y) \in D$ and making all other products equal to 0. A partial groupoid $G$ is said to be an *Z-presemigroup* if $G^0$ is a semigroup.

We introduce the following conditions on a partial groupoid $G$:

(A1) For all $x, y, z \in G$, if $(x, y) \in D$ and $(x \cdot y, z) \in D$, then $(y, z) \in D$, $(x, y \cdot z) \in D$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(A2) For all $x, y, z \in G$, if $(y, z) \in D$ and $(x, y \cdot z) \in D$, then $(x, y) \in D$, $(x \cdot y, z) \in D$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

**Theorem B** (Conrad [4], see also [3, Lemma 3.7]). *A partial groupoid $G$ is a Z-presemigroup if and only if it satisfies conditions* (A1) *and* (A2).

Clearly, a partial groupoid $G = (G, D, \mu)$ can be considered as a pair $M(G) = (G, R)$, where $R$ is the ternary relation on $G$ defined by $(x, y, z) \in R$ if and only if $(x, y) \in D$ and

$\mu(x, y) = z$. In the terminology of Malcev [21], $M(G)$ is the *model corresponding to G*. If we use a first-order language $L$ consisting of one ternary relation symbol, $p$ say, we can describe the class of all partial semigroups as a quasivariety (in the sense of [21, 22]).

The following theorem is a corollary to Theorem A and Proposition I.5.10 of [11]:

**Theorem C.** *The class of all partial semigroups is a quasivariety (in the sense of [21, 22]) defined by an infinite set P of quasi-identities of the form*

$$\forall x \forall y \forall x_1 \ldots \forall x_m [pv_1v_2x \wedge \cdots \wedge pv_{s-1}v_s y \to x = y],$$

*where* $v_1, \ldots, v_s \in \{x_1, \ldots, x_m\}$.

## 3. The class of all embeddable semigroup amalgams is not finitely axiomatizable

Let $\{S_i: i \in I\}$ be a family of semigroups, let $U$ be a subsemigroup of $S_i$ for all $i \in I$, and let $S_i \cap S_j = U$ for all $i, j \in I$ with $i \neq j$. The semigroup amalgam $\mathscr{A} = [\{S_i: i \in I\}; U]$ determines a partial groupoid $\mathscr{G}(\mathscr{A})$ on $\mathscr{G} = \bigcup \{S_i: i \in I\}$ in which a product of two elements is defined if and only if they both belong to the same $S_i$ and their product is then taken as their product in $S_i$ [3, Section 9.4]. A semigroup amalgam $\mathscr{A}$ is said to be *embeddable* in a semigroup if the partial groupoid $\mathscr{G}(\mathscr{A})$ is a partial semigroup.

**Theorem 1.** *The class of all embeddable semigroup amalgams is not first-order finitely axiomatizable.*

**Proof.** Let $P$ be the infinite set of quasi-identities defining the class of all partial semigroups and let $P_0$ be any finite subset of $P$. Let $n \geq 1$. We shall construct a semigroup amalgam $\mathscr{A}_n = [S_n, T_n; U_n]$ and a partial groupoid $\mathscr{G}_n = \mathscr{G}(\mathscr{A}_n)$ such that $M(\mathscr{G}_n)$, that is, the model corresponding to $\mathscr{G}_n$, satisfies $P_0$ but not $P$. From this it follows that the class of all embeddable semigroup amalgams is not first-order finitely axiomatizable. We refer the reader to [2] for the fundamentals of first-order theories.

On the set $G_S = \{s, s_1, \ldots, s_n, s'_1, \ldots, s'_{n-1}, u_1, \ldots, u_{n+1}, v_1, \ldots, v_n\}$ we define a partial groupoid $G_S$ by the multiplications

$$u_1 s_1 = s, \qquad v_n s_n = u_{n+1}, \qquad v_i s_i = u_{i+1} s_{i+1} = s'_i, \quad i = 1, \ldots, n-1.$$

On the set $G_T = \{t, t_1, \ldots, t_n, t'_1, \ldots, t'_{n-1}, u_1, \ldots, u_{n+1}, v_1, \ldots, v_n\}$ we define a partial groupoid $G_T$ by the multiplications

$$t_1 v_1 = u_1, \qquad t_n u_{n+1} = t, \qquad t_i u_{i+1} = t_{i+1} v_{i+1} = t'_i, \quad i = 1, \ldots, n-1.$$

By using Theorem B, one can easily verify that the partial groupoids $G_S$ and $G_T$ are Z-presemigroups. Denote $S_n = G_S^0$ and $T_n = G_T^0$. Clearly, the zero semigroup on $U_n = \{0, u_1, \ldots, u_{n+1}, v_1, \ldots, v_n\}$ is a subsemigroup of $S_n$ and $T_n$ such that

$S_n \cap T_n = U_n$. Denote $\mathscr{G}(\mathscr{A}_n)$ by $\mathscr{G}_n$, $\theta_0(\mathscr{G}_n)$ by $\theta_0$, and $\theta(\mathscr{G}_n)$ by $\theta$. One can easily see that

$$s_i\theta = \{s_i\},\ t_i\theta = \{t_i\},\ v_i\theta = \{v_i\}, \quad i = 1, \ldots, n,$$

$$s_j'\theta = \{s_j', v_js_j, u_{j+1}s_{j+1}\},\ t_j'\theta = \{t_j', t_ju_{j+1}, t_{j+1}v_{j+1}\},$$

$$j = 1, \ldots, n-1,\ u_1\theta = \{u_1, t_1v_1\},\ u_{n+1}\theta = \{u_{n+1}, v_ns_n\},$$

$$u_k\theta = \{u_k\}, \quad k = 2, \ldots, n.$$

The $\theta$-class of 0 contains exactly one element of $\mathscr{G}_n$, namely 0. Hence each of the above $\theta$-classes contains exactly one element of $\mathscr{G}_n$. But $s$ and $t$ belong to the same $\theta$-class. The sequence of elementary $\theta_0$-transitions of minimal length from $s$ to $t$ has the form

$$s \to u_1s_1 \to t_1v_1s_1 \to t_1s_1' \to t_1u_2s_2 \to t_1's_2 \to \cdots \to t_{n-1}'s_n \to t_nv_ns_n \to t_nu_{n+1} \to t.$$

The above sequence (and hence the sequence of minimal length from $t$ to $s$) has length $4n$.

Since $s$ and $t$ belong to the same $\theta$-class, by Theorem A, the partial groupoid $\mathscr{G}_n$ is not a partial semigroup. Hence $M(\mathscr{G}_n)$ does not satisfy $P$. For each quasi-identity $\psi \in P$ we denote by $l_p(\psi)$ the number of occurrences of the symbol $p$ in $\psi$. Let $l_p(\psi) = n_\psi$, $\psi \in P_0$. Choose $n \geqslant n_\psi$, for all $\psi \in P_0$. Then $M(\mathscr{G}_n)$ satisfies $P_0$ but not $P$. This completes the proof of Theorem 1. □

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